How to Add a Non-Integer Number of Terms, and How to Produce Unusual Infinite Summations

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Abstract. Sums of the form $\sum_{\nu=1}^{x} f(\nu)$ are defined traditionally only when the number of terms x is a positive integer or ∞. We propose a natural way to extend this definition to the case when x is a (rather arbitrary) real or complex number ("fractional sums"). This generalizes known special cases like the interpolation of the factorial by the Γ function, or Euler's little-known formula

$$\sum_{\nu=1}^{-1/2} \frac{1}{\nu} = -2\ln 2 \ .$$

After giving the fundamental definition, we generalize several algebraic identities (such as the geometric series) to the case with a non-integer number of terms.

We use these ideas to derive a number of unusual infinite sums, products and limits, such as

$$\lim_{n\to\infty}\left((2n)^{-\frac{n^2}{2}-\frac{n}{4}} e^{-\frac{n}{8}} \prod_{\nu=1}^{2n} \nu^{(-1)^\nu \frac{\nu^2}{4}} \right) = e^{\frac{7\zeta(3)}{16\pi^2}} \ .$$

Contents

1. Introduction

In this note, we determine a number of unusual infinite sums, products, or limits, such as

$$2\sum_{\nu\geq 1 \text{ odd}} \frac{1+\frac{1}{3^s}+\frac{1}{5^s}+...++\frac{1}{\nu^s}}{\nu^s} = \zeta(2s)(1-2^{-2s}) + \left(\zeta(s)(1-2^{-s})\right)^2 \tag{1}$$

for every real $s>1$ (using the Riemann ζ function), or, for every $a>0$,

$$\prod_{\nu=1}^{\infty} \frac{1}{e}\left(1+\frac{1}{a\nu}\right)^{a\nu+\frac{1}{2}} \tag{2}$$

$$= \sqrt{\frac{\Gamma(1+\frac{1}{a})}{2\pi}} \exp\left[\frac{1}{2}\left(1+\frac{1}{a}\right) - a\left(\zeta'\left(-1,1+\frac{1}{a}\right)-\zeta'(-1)\right)\right] .$$

Examples of limits include

$$\lim_{n\to\infty}\left((2n)^{-\frac{n^2}{2}-\frac{n}{4}} e^{-\frac{n}{8}} \prod_{\nu=1}^{2n} \nu^{(-1)^\nu \frac{\nu^2}{4}}\right) = e^{\frac{7\zeta(3)}{16\pi^2}} \tag{3}$$

or

$$\lim_{n\to\infty}\left[-\frac{5}{16}\ln n \ln(n!) + \frac{1}{16}\ln(n+2)\ln((n+2)!) \right. \tag{4}$$

$$\left. -\frac{1}{4}\ln(n+1)\ln((n+1)!) + \sum_{\nu=1}^{2n}(-1)^v \ln\frac{\nu}{2}\ln\left(\left(\frac{\nu}{2}\right)!\right)\right]$$

$$= \frac{\gamma^2}{4}+\frac{\gamma_1}{2}-\frac{\pi^2}{48}+\frac{\ln^2 2}{2}-\frac{\ln^2\pi}{8} ,$$

where γ and γ_1 are the Euler-Mascheroni and Stieltjes constants

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...\frac{1}{n}-\ln n\right) = .577215\ldots , \tag{5}$$

$$\gamma_1 = \lim_{n\to\infty}\left(\frac{\ln 1}{1}+\frac{\ln 2}{2}+...+\frac{\ln n}{n}-\frac{\ln^2 n}{2}\right) = -.072815\ldots .$$

We try to prove (or sketch the proofs of) all these identities in this note. At the end, we speculate about a short and unusual proof of the following identity from [**GIZ**] (for $b\in\mathbb{R}$)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\sqrt{b^2+\pi^2n^2} = \frac{\pi^2}{4}\left(\frac{\sin b}{b}-\frac{\cos b}{3}\right) . \tag{6}$$

These identities were derived by a method to evaluate sums in which the number of terms is no longer an integer, but almost any real or complex number. While this method has been mentioned by Euler or Ramanujan, we are not aware of any systematic attempts to work it out.

At the end of this note, we give a number of suggestions how to produce further identities of a similar kind.

In the remainder of this introduction, we give a heuristic derivation for our definition of "fractional sums"; a formal definition follows in Section 2.

Our goal is to make sense of sums $\sum_{\nu=1}^{x} f(\nu)$ for non-integer values of x. Our starting point is what we call the "continued summation identity"

$$\sum_{\nu=a}^{b} f(\nu) + \sum_{\nu=b+1}^{c} f(\nu) = \sum_{\nu=a}^{c} f(\nu) \tag{7}$$

which holds for integer values of a, b, c. Any extension of summation to non-integer numbers of terms should respect this identity. For $x \in \mathbb{C}$ and $n \in \mathbb{N}$ we should therefore have

$$\mathop{\sum\!\!\!\!\!\!\!\!\rightarrow}_{\nu=1}^{n+x} = \sum_{\nu=1}^{n} + \mathop{\sum\!\!\!\!\!\!\!\!\rightarrow}_{\nu=n+1}^{n+x} = \mathop{\sum\!\!\!\!\!\!\!\!\rightarrow}_{\nu=1}^{x} + \sum_{\nu=x+1}^{x+n} ; \tag{8}$$

here and in the following we have used the symbol $\sum\!\!\!\!\!\!\!\!\rightarrow$ for summations with non-integer numbers of terms (which will formally be defined in Section 2). Note that the last sum has non-integer summation boundaries, but the number of terms is still an integer: $\sum_{\nu=x+1}^{x+n} f(\nu) = \sum_{\nu=1}^{n} f(\nu + x)$.

Rearranging terms, we can rewrite this as

$$\mathop{\sum\!\!\!\!\!\!\!\!\rightarrow}_{\nu=1}^{x} f(\nu) = \sum_{\nu=1}^{n} \Big(f(\nu) - f(\nu + x) \Big) + \mathop{\sum\!\!\!\!\!\!\!\!\rightarrow}_{\nu=n+1}^{n+x} f(\nu) \ . \tag{9}$$

We want to define the left hand side in terms of the right hand side whenever we can make sense of the last sum (which still involves a summation over x terms). Note that this expression is valid for arbitrary $n \in \mathbb{N}$, so all we need is a useful limit as $n \to \infty$.

If for example $f(\nu) \to 0$ as $\nu \to +\infty$, then we can argue heuristically that the sum $\mathop{\sum\!\!\!\!\!\!\!\!\rightarrow}_{\nu=n+1}^{n+x} f(\nu)$ should also tend to 0 as $n \to \infty$, because the number x of terms is unchanged. So in this case, we have found a definition for the fractional sum:

$$\sum_{\nu=1}^{x} f(\nu) := \sum_{\nu=1}^{\infty} \Big(f(\nu) - f(\nu + x) \Big) \ . \tag{10}$$

Similarly, in cases such as $f(\nu) = \log(\nu)$, the function f can be approximated by constants over regions of bounded diameters: for every bounded domain W and every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ there is a $C_n \in \mathbb{C}$ for which $|f(z+n) - C_n| < \varepsilon$ uniformly for $z \in W$. It thus makes sense to estimate $\sum_{\nu=n+1}^{n+x} f(\nu) \approx x f(n)$ (the approximating constants change with n, but the quality of the approximation improves for large n). Our Definition 2 will be a generalization of this idea for approximation by arbitrary polynomials: after all, for polynomials we have formulas like

$$\sum_{\nu=1}^{x} \nu = \frac{x(x+1)}{2}$$

which can be generalized without change to arbitrary $x \in \mathbb{C}$.

This note is a brief introduction to the ideas and basic results. More details, as well as the proofs which we omitted here, can be found in a more detailed manuscript [**MS**].

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2. The Fundamental Definition

We begin by a natural definition for polynomials.

Definition 1 (Fractional Sums for Polynomials).
For every polynomial $p : \mathbb{C} \to \mathbb{C}$, we define

$$\sum_{\nu=1}^{x} p(\nu) := P(x) \ ,$$

where P denotes the unique polynomial that agrees with the classical value of the sum for every positive integer x. Moreover, for arbitrary $a, b \in \mathbb{C}$, we define

$$\sum_{\nu=a}^{b} p(\nu) := \sum_{\nu=1}^{b-a+1} p(\nu + a - 1) = P(b) - P(a-1) \ .$$

Now comes our general definition of fractional sums. It uses the approximating polynomials as well as their fractional sums as just defined. The cases discussed in the introduction are those when all $p_n \equiv 0$ (i.e. $\sigma = -\infty$) or when they are constant ($\sigma = 0$).

Definition 2 (Fractional Summable Functions).
Let $U \subset \mathbb{C}$ and $\sigma \in \mathbb{N} \cup \{-\infty\}$. A function $f \colon U \to \mathbb{C}$ will be called right summable of degree σ *if the following conditions are satisfied:*

- *$U + 1 \subset U$, where $U + 1 = \{u + 1 \colon u \in U\}$;*
- *there exists a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ of fixed degree σ such that*
  $$|f(n+x) - p_n(n+x)| \longrightarrow 0 \quad \text{as } n \to +\infty$$
  for all x in every bounded subset of U ;
- *for every $a, b+1 \in U$, the limit*
  $$\lim_{n\to\infty} \left(\sum_{\nu=n+a}^{n+b} p_n(\nu) + \sum_{\nu=1}^{n} \Big(f(\nu + a - 1) - f(\nu + b) \Big) \right)$$
  exists.

In this case, we will use the notation

$$\sum_{\nu=a}^{b} f(\nu) \qquad \text{or briefly} \qquad \sum_{a}^{b} f$$

for this limit. Moreover, we can define fractional products by

$$\prod_{\nu=a}^{b} f(\nu) := \exp\left(\sum_{\nu=a}^{b} \ln f(\nu) \right) ,$$

whenever $\ln f$ is right summable.

Remark.

- In the limit, $n \in \mathbb{N}$ is always taken to be an integer.
- The value of the sum is independent of the choice of the approximating polynomials p_n.
- If $b - a \in \mathbb{N}$, then the limit exists and agrees with the classical value of the sum.
- It is easy to extend the heuristic argument in (8) to the case of functions covered by this definition.
- This definition only covers a certain class of functions f which we call the set of approximately polynomial functions. As we will see, this class of functions is large enough for many

applications. If one tries to consider a larger class of functions (say, any analytic function), certain serious problems arise.

The following basic properties follow immediately from the definition:

THEOREM 3 (Basic Properties of Fractional Sums).
Fractional sums have the following properties for arbitrary $a, b, c, d \in \mathbb{C}$:

- *Linearity:* $c \sum_{\nu=a}^{b} f(\nu) + d \sum_{\nu=a}^{b} g(\nu) = \sum_{\nu=a}^{b} \Big(cf(\nu) + dg(\nu) \Big)$,
- *Continued Summation:* $\sum_{\nu=a}^{b} f(\nu) + \sum_{\nu=b+1}^{c} f(\nu) = \sum_{\nu=a}^{c} f(\nu)$,
- *Index Shifting:* $\sum_{\nu=a}^{b} f(\nu + c) = \sum_{\nu=a+c}^{b+c} f(\nu)$,

whenever two of the three fractional sums (in the last case: one of the two sums) exist.

Note also that $\sum_{x}^{x-1} f = 0$ and $\sum_{x}^{x} f = f(x)$, and more generally

$$\sum_{x}^{x+n} f = f(x) + f(x+1) + \ldots + f(x+n)$$

for $n \in \mathbb{N}$.

3. Basic Algebraic Identities

In this section, we give a short summary on how classical sum identities generalize to the case of fractional sums.

In the following theorem, we use the usual conventions $0^0 = 1$ and $\binom{c}{\nu} = \frac{\Gamma(c+1)}{\Gamma(c-\nu+1)\Gamma(\nu+1)}$ (compare Theorem 6).

THEOREM 4 (The Geometric and the Binomial Series).
For every $x \in \mathbb{C}$, $0 \le q < 1$, $c \in \mathbb{C} \setminus \{-1, -2, -3, ...\}$ and $z \in \mathbb{C}$ with $|z| < 1$, we have

$$\sum_{\nu=0}^{x} q^\nu = \frac{1 - q^{x+1}}{1 - q} ,$$

$$(1+z)^c = \sum_{\nu=0}^{c} \binom{c}{\nu} z^\nu .$$

Proof. In both cases, we can take $p_n \equiv 0$ as approximating polynomials. We use the definition (compare Equation (10)) and calculate

$$\sum_{\nu=1}^{x+1} q^{\nu-1} = \sum_{\nu=1}^{\infty} \left(q^{\nu-1} - q^{\nu+x}\right) = (1-q^{x+1}) \sum_{\nu=1}^{\infty} q^{\nu-1} = \frac{1-q^{x+1}}{1-q} \ ,$$

$$1 + \sum_{\nu=1}^{c} \binom{c}{\nu} z^\nu = 1 + \sum_{\nu=1}^{\infty} \left(\binom{c}{\nu} z^\nu - \binom{c}{\nu+c} z^{\nu+c} \right) = \sum_{\nu=0}^{\infty} \binom{c}{\nu} z^\nu \ ,$$

where the second part of the sum vanishes, since $\binom{c}{\nu+c} = \frac{\Gamma(c+1)}{\Gamma(\nu+c+1)\Gamma(1-\nu)} = 0$ for every $\nu \in \mathbb{N} \setminus \{0\}$. □

Remark. There is a generalization of these formulas to the case when $q > 1$ and $|z| > 1$ which involves a "left sum" $\overleftarrow{\sum}_{\nu=1}^{x}$.

Proposition 5 (The Squaring Formula).
For every right summable function $f : U \to \mathbb{C}$, we have

$$\left[\sum_{\nu=1}^{x} f(\nu) \right]^2 = \sum_{\nu=1}^{x} \left[f^2(\nu) + 2f(\nu) \sum_{k=1}^{\nu-1} f(k) \right] = \sum_{\nu=1}^{x} \left[-f^2(\nu) + 2f(\nu) \sum_{k=1}^{\nu} f(k) \right]$$

whenever $\left(\sum_{\nu=1}^{x} f(\nu)\right)^2$ is an approximate polynomial in x.

There is not enough space for the proof in this note. The squaring formula will serve for two purposes: It shows that also identities with more complicated structure nicely generalize to the fractional case, and it will be useful for proving Equation (4).

4. Fractional Sums and Special Functions

In this section, we will consider simple examples of fractional sums that turn out to be related to special functions. The first example shows that our definition is consistent with the classical interpolation of the factorial by the Gamma function:

Theorem 6 (The Gamma Function).
For every $x \in \mathbb{C} \setminus \{-1, -2, -3, ...\}$, we have

$$\prod_{\nu=1}^{x} \nu = \lim_{n\to\infty} \left(n^x \prod_{\nu=1}^{n} \frac{\nu}{\nu+x} \right) = \Gamma(x+1) \ .$$

PROOF. Consider the fractional sum $\sum_{\nu=1}^{x} \ln \nu$. The logarithm is "approximately constant": we can take $p_n(x) := \ln n$ as approximating polynomials according to Definition 2, since $|\ln(n+x) - \ln n| \to 0$ as $n \to \infty$.

By Definition 1, a constant sum gives $\sum_{\nu=n+1}^{n+x} \ln n = x \ln n$, so

$$\sum_{\nu=1}^{x} \ln \nu = \lim_{n\to\infty} \left(x \ln n + \sum_{\nu=1}^{n} \Big(\ln \nu - \ln(\nu + x) \Big) \right) .$$

According to Definition 2, we have

$$\prod_{\nu=1}^{x} \nu = \exp\left(\sum_{\nu=1}^{x} \ln \nu \right) = \lim_{n\to\infty} \left(n^x \prod_{\nu=1}^{n} \frac{\nu}{\nu + x} \right) ,$$

and it is well-known (Gauss' Formula) that this equals $\Gamma(x+1)$. □

The next example relates the fractional sum of monomials to the Riemann and Hurwitz Zeta functions, defined as the analytic continuations of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$ respectively.

PROPOSITION 7 (The Riemann and Hurwitz Zeta Functions). *For every $x \in \mathbb{C} \setminus \{-1,-2,-3,...\}$, $z \in \mathbb{C} \setminus \{-1\}$ and $b \in \mathbb{N}$, we have*

$$\sum_{\nu=1}^{x} \nu^z = \zeta(-z) - \zeta(-z, x+1) , \tag{11}$$

$$\sum_{\nu=1}^{-\frac{1}{2}} \nu^z = \left(2 - 2^{-z}\right) \zeta(-z) , \tag{12}$$

$$\sum_{\nu=1}^{x} \nu^z \left(\ln \nu\right)^b = (-1)^b \left(\zeta^{(b)}(-z) - \zeta^{(b)}(-z, x+1)\right) , \tag{13}$$

where $\zeta^{(b)} = (\partial/\partial z)^b \zeta$.

PROOF. Equation (12) is a special case of Equation (11). If $\mathrm{Re}(z) < -1$, we can again take $p_n \equiv 0$ as approximating polynomials, and the definition reads

$$\begin{aligned} \sum_{\nu=1}^{x} \nu^z &= \sum_{\nu=1}^{\infty} (\nu^z - (\nu+x)^z) = \sum_{\nu=1}^{\infty} \nu^z - \sum_{\nu=0}^{\infty} (\nu + x + 1)^z \\ &= \zeta(-z) - \zeta(-z, x+1) . \end{aligned}$$

The case that $\mathrm{Re}(z) > -1$ is rather nontrivial. Note that Equation (13) follows by formal differentiation of Equation (11) b times with respect to z. □

REMARK. A special case for $z = -1$ is

$$\sum_{\nu=1}^{-\frac{1}{2}} \frac{1}{\nu} = -2\ln 2$$

which was noticed already by Euler [**E**].

5. Fractional Sum Identities

The relation between fractional sums and special functions allows one to find exotic analytic expressions for certain sums which involve well-known constants. This section shows two examples.

EXAMPLE 8 (Power Tower Product).
We have the identity

$$\prod_{n=1}^{-\frac{1}{2}} n^{n^2} = e^{\frac{7\zeta(3)}{16\pi^2}} \ . \tag{14}$$

PROOF. Using (13) from Proposition 7, as well as

$$\zeta\left(s, \frac{1}{2}\right) = \sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right)^{-s} = 2^s \sum_{n=1}^{\infty} (2n-1)^{-s} = (2^s - 1)\zeta(s) \ ,$$

we can calculate

$$\sum_{n=1}^{-\frac{1}{2}} n^2 \ln n = -\left(\zeta'(-2) - \zeta'\left(-2, \frac{1}{2}\right)\right) = \frac{d}{dz}\left(2 - 2^{-z}\right)\zeta(-z)\Big|_{z=2} \ .$$

To evaluate this expression, we use that $\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}$, which follows by differentiating the functional equation of the Riemann ζ function, for example in the form [**AS**, (23.2.6)]. We obtain $\sum n^2 \ln n = \frac{7\zeta(3)}{16\pi^2}$. The claim follows by exponentiation. □

REMARK. The function $n^2 \ln n$ is approximately polynomial of degree $\sigma = 2$. Expanding (14) using the definition of fractional products and simplifying yields Equation (3) as a classical representation of this result.

The next example involves the Euler-Mascheroni constant γ and one of the so-called Stieltjes constants γ_1, defined by Equation (5).

Example 9 (Stieltjes Constants Identity). *We have*

$$\sum_{n=1}^{-\frac{1}{2}} \ln n \ln(n!) = \frac{\gamma^2}{4} + \frac{\gamma_1}{2} - \frac{\pi^2}{48} + \frac{\ln^2 2}{2} - \frac{\ln^2 \pi}{8} \ .$$

Proof. We start with the well-known identity $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Recalling Theorem 6 and taking logarithms, we then have

$$\sum_{\nu=1}^{-\frac{1}{2}} \ln \nu = \frac{1}{2} \ln \pi \ .$$

Now we use the squaring formula from Proposition 5; rewrite it to get

$$\left[\sum_{\nu=1}^{-\frac{1}{2}} \ln \nu \right]^2 = - \sum_{\nu=1}^{-\frac{1}{2}} \ln^2 \nu + 2 \sum_{\nu=1}^{-\frac{1}{2}} \left[\ln \nu \sum_{k=1}^{\nu} \ln k \right] \ .$$

According to Equation (13) from Proposition 7, we can also write

$$\frac{1}{4} \ln^2 \pi = -\frac{d^2}{dz^2} \sum_{\nu=1}^{-\frac{1}{2}} \nu^z|_{z=0} + 2 \sum_{\nu=1}^{-\frac{1}{2}} \ln \nu \ln(\nu!) \ .$$

The second sum is again related to derivatives of the Riemann Zeta Function by

$$\frac{d^2}{dz^2} \sum_{\nu=1}^{-\frac{1}{2}} \nu^z|_{z=0} = \frac{d^2}{dz^2} \left(2 - 2^{-z}\right) \zeta(-z)|_{z=0} \ .$$

This is easy to evaluate using the values $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ [**AAR**, Corollary 1.3.3] and $\zeta''(0) = \frac{\gamma^2}{2} - \frac{\pi^2}{24} - \frac{1}{2}\ln^2(2\pi) + \gamma_1$: this follows again by differentiating the functional equation, using the identity $\frac{d}{ds}\left(\zeta(s) + \frac{1}{1-s}\right)|_{s=1} = -\gamma_1$ [**AS**, (23.2.5)]. □

Remark. The function $\ln n \ln(n!)$ is approximately polynomial of degree $\sigma = 2$. Similarly as in the previous example, one can find a classical representation of this result by resolving the definition of the fractional sum. Here, we get Equation (4).

6. Classical Identities

In the previous part, it has already been shown that fractional sum identities can be written as (sometimes lengthy) classical limit formulas. In this last section, some more examples will be discussed, to show in what way fractional sums can be used to derive classical infinite sum identities.

Example 10 (Partial Euler Sum).
For every $s > 1$, we have the identity

$$2 \sum_{\nu \ odd} \frac{1 + \frac{1}{3^s} + \frac{1}{5^s} + ... + + \frac{1}{\nu^s}}{\nu^s} = \zeta(2s)(1 - 2^{-2s}) + \big(\zeta(s)(1 - 2^{-s})\big)^2 \ .$$

Proof. We start with the squaring formula from Proposition 5 for the function ν^{-s}:

$$\left[\sum_{\nu=1}^{x} \nu^{-s} \right]^2 = - \sum_{\nu=1}^{x} \nu^{-2s} + 2 \sum_{\nu=1}^{x} \nu^{-s} \sum_{k=1}^{\nu} k^{-s} \ .$$

We first consider the case $x = \infty$, and get

$$\zeta^2(s) = -\zeta(2s) + 2 \sum_{\nu=1}^{\infty} \nu^{-s} \sum_{k=1}^{\nu} k^{-s} \ . \tag{15}$$

Recalling Equation (12) from Proposition 7, we have for $x = -\frac{1}{2}$

$$(2 - 2^s)^2 \zeta^2(s) = - \left(2 - 2^{2s}\right) \zeta(2s) + 2 \sum_{\nu=1}^{-\frac{1}{2}} \nu^{-s} \sum_{k=1}^{\nu} k^{-s} \ . \tag{16}$$

Expanding the definition of fractional sums, and then using Equation (15) and continued summation, we get

$$\begin{aligned} \sum_{\nu=1}^{-\frac{1}{2}} \nu^{-s} \sum_{k=1}^{\nu} k^{-s} &= \sum_{\nu=1}^{\infty} \left(\nu^{-s} \sum_{k=1}^{\nu} k^{-s} - \left(\nu - \frac{1}{2}\right)^{-s} \sum_{k=1}^{\nu - \frac{1}{2}} k^{-s} \right) \\ &= \frac{\zeta^2(s) + \zeta(2s)}{2} - \sum_{\nu=1}^{\infty} \left(\nu - \frac{1}{2}\right)^{-s} \left(\sum_{k=1}^{-\frac{1}{2}} k^{-s} + \sum_{k=\frac{1}{2}}^{\nu - \frac{1}{2}} k^{-s} \right) \\ &= \frac{1}{2}\zeta^2(s) + \frac{1}{2}\zeta(2s) - 2^s \left(2 - 2^s\right) \zeta(s) \sum_{\nu=1}^{\infty} (2\nu - 1)^{-s} \\ &\quad -2^{2s} \sum_{\nu=1}^{\infty} (2\nu - 1)^{-s} \sum_{k=1}^{\nu} (2k - 1)^{-s} \ . \end{aligned}$$

Using that $\sum_{\nu=1}^{\infty} (2\nu - 1)^{-s} = (1 - 2^{-s})\zeta(s)$, the claim follows from Equation (16). □

PROPOSITION 11 (Product and Polygamma Identity).
For the polygamma function $\psi_2(z) = \frac{d^3}{dz^3} \ln \Gamma(z)$ *and every* $a > 0$*, we have the identities*

$$\sum_{\nu=1}^{\infty} (-1)^{\nu+1} \psi_2 \left(\frac{\nu}{10} + \frac{1}{2} \right) = -7\zeta(3) + \frac{50}{3}\pi^2 - \frac{36}{5}\sqrt{5 - 2\sqrt{5}}\pi^3 \ ,$$

and

$$\prod_{\nu=1}^{\infty} \frac{1}{e} \left(1 + \frac{1}{a\nu} \right)^{a\nu + \frac{1}{2}}$$

$$= \sqrt{\frac{\Gamma(1+\frac{1}{a})}{2\pi}} \exp \left[\frac{1}{2} \left(1 + \frac{1}{a} \right) - a \left(\zeta' \left(-1, 1 + \frac{1}{a} \right) - \zeta'(-1) \right) \right] \ .$$

REMARK. The special case $a = 3$ is due to Gosper ([**M**, (93)]).

PROOF. The first identity is proved in a similar way as Example 10, just be rewriting it as a double sum which can be computed using the squaring formula from Proposition 5. The proof of the infinite product identity is similar to that of the power tower product from Example 8: By taking logarithms, the product can be split into several fractional sums with $1/a$ terms, which can then be analytically expressed in terms of the Γ function and derivatives of the Riemann ζ Function, by using Theorem 6 and Proposition 7 respectively. □

7. Perspectives, or How to Do it Yourself

The paper "On some strange summation formulas" [**GIZ**] contains formula (17) below. There might possibly be a very short proof for this identity using fractional sums. The only problem is that there is one single step (indicated by the question mark) of which we do not know whether it is justified: it is basically an interchange of a fractional sum and an infinite series.

Nevertheless, we give this calculation as a speculation, just to show that it is tempting to have a closer look at what else might still be possible.

SPECULATION (A Series by Gosper).
For every $b \in \mathbb{R}$*, we have the identity*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \sqrt{b^2 + \pi^2 n^2} = \frac{\pi^2}{4} \left(\frac{\sin b}{b} - \frac{\cos b}{3} \right) \ . \tag{17}$$

"PROOF". We start from a fractional sum with $-\frac{1}{2}$ terms. When evaluating it using the definition (with approximating polynomials $p_n \equiv 0$), it turns out to be equal to the desired infinite series:

$$\frac{1}{4}\sum_{n=1}^{-\frac{1}{2}} \frac{1}{n^2}\cos\sqrt{b^2+4\pi^2n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\sqrt{b^2+\pi^2n^2} =: S(b)\ .$$

We call the value of the infinite series $S(b)$. The function $f(n) := \frac{1}{n^2}\cos\sqrt{b^2+4\pi^2n^2}$ is holomorphic in the entire complex plane, except for a pole at $n=0$, so we can develop it into a power series; since f and cos are even, we get

$$S(b) = \frac{1}{4}\sum_{n=1}^{-\frac{1}{2}}\left(\frac{\cos b}{n^2} - \frac{2\pi^2\sin b}{b} + c_2n^2 + c_4n^4 + \dots\right)\ .$$

The next step is critical: We apply the fractional sum term by term. Unfortunately, it is not clear that this manipulation is justified.

$$S(b) \overset{?}{=} \frac{1}{4}\left[\sum_{n=1}^{-\frac{1}{2}}\frac{\cos b}{n^2} - \frac{2\pi^2\sin b}{b}\sum_{n=1}^{-\frac{1}{2}}1 + c_2\sum_{n=1}^{-\frac{1}{2}}n^2 + c_4\sum_{n=1}^{-\frac{1}{2}}n^4 + \dots\right]$$

Using Equation (12) from Proposition 7, we know that

$$\sum_{n=1}^{-\frac{1}{2}}\frac{1}{n^2} = -\frac{\pi^2}{3}\ ,\quad \sum_{n=1}^{-\frac{1}{2}}1 = -\frac{1}{2}\ ,\quad \sum_{n=1}^{-\frac{1}{2}}n^{2k} = 0\ ,$$

so all the fractional sums $\sum_{n=1}^{-\frac{1}{2}} n^{2k}$ disappear for positive integers k (because of the trivial zeroes of the Riemann zeta function), and only finitely many terms survive:

$$S(b) = \frac{1}{4}\cos b\left(-\frac{\pi^2}{3}\right) - \frac{2\pi^2\sin b}{4b}\left(-\frac{1}{2}\right) = \frac{\pi^2}{4}\left(\frac{\sin b}{b} - \frac{\cos b}{3}\right)\ .$$

□

REMARK. There are other identities in [**GIZ**] which can be reproduced in a similar way, using other power series identities rather than summing from 1 to $-\frac{1}{2}$. For example, for every $k \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

$$\sum_{n=z}^{-z}\frac{1}{n} = \pi\cot\pi z\quad ,\quad \sum_{n=z}^{-z}n^{2k+1} = 0\ .$$

Although this works in the case of identities from [**GIZ**] and others like $\sum \sin(an)/n^k$ or $\sum e^{an}/n^k$, it already fails for sums like $\sum e^{-an^2}$, so a lot more work has to be done to make clear under what conditions this method can be applied.

What we have shown here can be seen as a typical example for some kind of general method to compute infinite series; it works as follows:

Fractional Sums: The Systematic Method

- Select a limit, infinite sum or product you want to compute.
- Rewrite it as a fractional sum or product.
- Use fractional sum identities (like, for example, Proposition 7, Theorem 6, the squaring formula or the continued summation identity) to simplify.
- Rewrite the result in a classical way, by resolving its definition.

The problem is that this works only in a few special cases. For example, it has produced the Gosper series in Equations (2) and (6), but we have not found many other examples. Here is a second possibility:

Fractional Sums: The Creative Method

- Select a classical finite identity for summing functions f; the more complicated, the better. (Examples include the squaring formula in Proposition 5 and its extensions to higher powers, series multiplication, Abel's partial summation formula and others.)
- Prove that it remains valid for the fractional case.
- Specialize the function f, for example $f(\nu) = \ln \nu$.
- Simplify.
- Rewrite the result in a classical way, by resolving its definition.

The problem with this method is of course that it often produces trivial or useless formulas. Nevertheless, it is a do-it-yourself-method for creating curious identities. This way, we have found the identities in Examples 9 and 10.

Here is another example: look at the quadratic array of terms

$$\begin{array}{cccccccc} & f(1) & + & f(2) & + & \dots & + & f(x) \\ + & f(x+1) & + & f(x+2) & + & \dots & + & f(2x) \\ + & f(2x+1) & + & f(2x+2) & + & \dots & + & f(3x) \\ + & \dots & + & \dots & + & \dots & + & \dots \\ + & f((x-1)x+1) & + & f((x-1)x+2) & + & \dots & + & f(x^2) \end{array}$$

We can add all the terms or line by line, so

$$\sum_{\nu=1}^{x^2} f(\nu) = \sum_{k=1}^{x} f(k) + \sum_{k=x+1}^{2x} f(k) + \ldots + \sum_{k=(x-1)x+1}^{x^2} f(k) \ .$$

More generally, we can examine if

$$\sum_{\nu=1}^{x^2} f(\nu) = \sum_{\nu=1}^{x} \sum_{k=(\nu-1)x+1}^{\nu x} f(k)$$

holds for fractional sums (it turns out that it does). We can now proceed with the Creative Method using $f(\nu) = \ln \nu$, which gives us the possibility to write the partial sums as the logarithm of the factorial, and $x = \sqrt{2}$, which gives a classical sum on the left hand side. After simplification and exponentiation, one finally arrives at

$$\prod_{n=1}^{\sqrt{2}} \binom{n\sqrt{2}}{\sqrt{2}} = \frac{2}{\left(\sqrt{2}\,!\right)^{\sqrt{2}}} ,$$

where $\sqrt{2}\,! \equiv \Gamma(1+\sqrt{2})$.

Expanding the definition, it turns out that this identity is trivial, so the Creative Method was not successful in this particular case. Never mind: this method often does produce interesting identities. Try again!

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