Macroscopic Locality
and
the Tsirelson Bound

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Prelude

In 1969, John Clauser et al.\textsuperscript{(1)} obtained an inequality which, a particular set of correlations between two parties, must be satisfied in any classical theory (in the Bell language, in any local hidden variables theories). This inequality has maximum 2, in the absolute value. Eleven years later, in 1980, Boris Tsirelson\textsuperscript{(2)}, evaluated the inequality in the quantum case and, due to the Hermiticity of the operators and the Hilbert space structure of the quantum states, obtained $2\sqrt{2}$ as the maximum value of the inequality, in the absolute value sense; this number is called, the Tsirelson bound.

One of the questions that comes to mind is, why $2\sqrt{2}$, not another? Sandu Popescu and Daniel Rohrlich, in 1994\textsuperscript{(3)}, approached this question in a novel way and, considered what correlations will have been obtained if, one considers non-locality as the underlying principle and, by the no-signaling constraint on correlations. They discovered that, these two principles are not sufficient to obtain quantum correlations, i. e. which do not violate Tsirelson bound, on the contrary, they obtained the possibility of correlations that, were more stronger non-local than the quantum theory. So, by the fact that quantum theory is not obtainable, in principle, from the two axioms of no-signaling constraint and non-locality, people have started searching for some supplementary principles which, could rule out the unwanted correlations; actually some of these correlations are un-physical, for example in a world where Communication Complexity is not trivial\textsuperscript{(4)}.

The supplementary principles that people have suggested could be Communication Complexity\textsuperscript{(4)}, Information Causality\textsuperscript{(5)}, Macroscopic Locality\textsuperscript{(6)} and so on.

Here, we discuss first, the notion of macroscopic locality, second, we show that quantum correlations are macroscopically local and, finally, macroscopic locality implies the Tsirelson bound.
1. Introduction

Simply, macroscopic locality is the fact that, when non–local correlations in microscopic experiments have been brought to the macroscopic scale, the resultant distribution must be governed by classical physics, i.e. a local hidden variable theory, otherwise, these microscopic correlations are un–physical.

First, we look at the usual case of microscopic experiment of non–locality.

Two distant observers, say, Alice and Bob, observe an event in the middle of their separation space which, produces two particles, one of them is received by Alice and the other by Bob. Both Alice and Bob have $s$ possibilities to perform a random measurement on their particles, $X$ and $Y$, respectively, and after the interactions, the particles will be received in one of the detectors in their laboratories. The measurements which Alice could perform will run from 1 to $s$, and for Bob goes form $s + 1$ to $2s$.

By repetition of the experiments many times, they could compare their measurement outcomes, and by statistical inference, they could deduce the probabilities $P(a, b)$, which means the probability that, Alice will obtain $a$ as her
outcome by performing some particular measurement, e. g. \( X \) and, Bob will obtain \( b \) as his outcome by performing some particular measurement, e. g. \( Y \).

We suppose that, no – signaling principle holds,

\[
\sum_{a \in X} P(a, b) = \sum_{a \in X'} P(a, b),
\]

Where, \( X \neq X' \), and the same argument holds for Bob. It simply means that, one party, say Alice, cannot simultaneously learn about the choice of the measurement of the other party, say Bob.

Also, \( P(a, b) \) is not necessarily classical, in other words, there is not a common (global) distribution (at least in some cases), which, 

\[
P(a, b) = \sum_c P(..., c_{X-2}, c_{X-1}, a, c_{X+1}, ..., c_{Y-2}, c_{Y-1}, b, c_{Y+1}, ...),
\]

Where, \( c \) is some outcome.

By going to the macroscopic scale, if we assume that the resultant distributions must be governed by classical physics, i. e. a local hidden variable theory, as a result of this assumption, we could discard some of \( P(a, b) \) as unphysical which, do not give classical physics as a limit.

Now, we illustrate the macroscopic experiment.

Alice’s lab.

Bob’s lab.

Macroscopic distribution \( P(I_X, I_Y) \)
In the macroscopic scale, it is reasonable that, instead of a pair of particles, one considers \( N \) independent pairs, which the number of pairs, \( N \), is much greater than 1, i.e. the thermodynamic limit. Each party, now, will receive a beam of particles and, like the microscopic case, each of them could perform a random measurement on their own beam, but now, the outcomes will be the intensities of the divided beams emanated from the two original beams as a result of some measurements and, are recorded in all detectors. As an assumption, due to make a relationship with classical physics, we assume that, these beams are continuous fields, in other words, Alice and Bob and also their detectors in their labs, cannot resolve the beams to their constituent particles (also, this implies that, they cannot perform different measurements on different particles, they are eligible to perform same interactions on the whole beam). Hence, the resolution of their detectors should not be perfect, and also a very poor resolution always gives the same distribution equal to the mean value, i.e. \( N P(a) \). Because we are interested in observing intensity fluctuations, we assume that, the detectors work in such a precision which, one could observe the deviations from the mean value, of the order \( \sqrt{N} \), if so, the resultant distributions will be described by classical physics, i.e. a local hidden variable theory.

2. Macroscopic locality constraint

We denote Alice’s outcome, i.e. \( \bar{I}_X \), by the row vector \( (\bar{I}_X^1, \bar{I}_X^2, \bar{I}_X^3, \bar{I}_X^4, ..., \bar{I}_X^d) \), where, the lower index denotes the particular measurement and, the upper one denotes the number of detector in the Alice’s laboratory and, the same notation holds for Bob, i.e. \( \bar{I}_Y = (\bar{I}_Y^1, \bar{I}_Y^2, \bar{I}_Y^3, \bar{I}_Y^4, ..., \bar{I}_Y^d) \).

Where, \textit{bar} above the intensities indicate that, we are dealing with fluctuations. (For precise definition see section (3.1).)

Like the microscopic experiment, by the repetition of the experiments many times and comparing the measurement outcomes, they could evaluate marginal distributions which, are Gaussians (see section 3), for some particular measurements \( X \) and \( Y \) as, \( P(\bar{I}_X, \bar{I}_Y) \).
Classical physics is a local hidden variable, as a result, there must be a global probability density \( P(\bar{I}_1, \bar{I}_2, \bar{I}_3, \ldots, \bar{I}_{2s}) \) and, the marginals of this global probability density satisfy,

\[
P(\bar{I}_X, \bar{I}_Y) = \int (\prod_{Z\neq X,Y} d\bar{I}_Z) P(\bar{I}_1, \bar{I}_2, \bar{I}_3, \ldots, \bar{I}_{2s}), \quad (2.1)
\]

Where, this implies the Macroscopic Locality constraint.

### 3. Macroscopic locality as a necessary condition for quantum physics

We are in the place to show that, quantum correlations are macroscopically local.

In 3.1, we will show that, the intensity fluctuations will be governed by a Gaussian distribution and, obtain a necessary and sufficient condition for macroscopic locality. Then, in 3.2, we prove the equivalency of correlations that emerge from macroscopic locality and no – signaling constraint, by a previous introduced set of correlations which is a first approximation to the quantum correlations \( Q \), that is, \( Q \subset Q^1 \) and as a consequence, quantum correlations are macroscopically local.

#### 3.1 Necessary and sufficient condition for being macroscopically local

We suppose that Alice and Bob have done some particular measurements, say \( X \) and \( Y \) and, we define the Kronecker delta – like observables, \( d_i^a \) and \( d_i^b \) for Alice and Bob, respectively, which, if the \( i^{th} \) particle of Alice gives \( a \) as its outcome, \( d_i^a \) is 1, otherwise vanishes and, the same arguments hold for Bob and his observables \( d_i^b \). In the macroscopic experiment, the resultant intensities, for example for Alice, \( I^a \), would be proportional to the summation of the \( d_i^a \) for \( N \) particles and the same for Bob’s, \( I^b \),

\[
I^a \approx \sum_{i=1}^{N} d_i^a \quad (3.1)
\]

\[
I^b \approx \sum_{i=1}^{N} d_i^b \quad (3.2)
\]
But, these are not very satisfactory because, as we mentioned before, we are interested to observe the **intensity fluctuations** of the order $\sqrt{N}$, so it is reasonable to define,

$$d_{a_i}^a \equiv (d_{i}^a - P(a)) \frac{1}{\sqrt{N}} \quad (3.3)$$

It is clear that, $\langle d_{a_i}^a \rangle$ vanishes for all possible outcomes. Now, fluctuations are proportional to the summation of the above values for $N$ particles,

$$\bar{I}^a \approx \sum_{i=1}^{N} d_{a_i}^a \quad (3.4)$$

We can simply consider the proportionality constant 1 and, write,

$$\bar{I}^a = \sum_{i=1}^{N} d_{a_i}^a \quad (3.5)$$

The same argument gives $d_{a_i}^b$ and $\bar{I}^b$ for the observer Bob. We put these fluctuations, Alice’s and Bob’s, together and write,

$$\bar{I} = \left( \frac{\bar{I}^X}{\bar{I}^Y} \right), \quad (3.6)$$

For some fixed measurements, say $X$ and $Y$.

Now, we investigate the thermodynamic limit, i. e. $N \to \infty$, and show that $\bar{I}$ converges to a Gaussian distribution, with some specified parameters.

**Central limit theorem**

If $\tilde{x}$ is a random vector, $(x_1, x_2, \ldots x_k)^T$, with the expectation value,

$$\langle \tilde{x} \rangle = (\langle x_1 \rangle, \langle x_2 \rangle, \ldots, \langle x_k \rangle)^T, \quad (3.7)$$

And the covariance matrix, symmetric and semi - definite positive,

$$Cov(\tilde{x}) = \langle (\tilde{x} - \langle \tilde{x} \rangle)(\tilde{x} - \langle \tilde{x} \rangle)^T \rangle, \quad (3.8)$$

Then,

$$\overline{S}_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\tilde{x}_i - \langle \tilde{x}_i \rangle) \quad (3.9)$$
as \( N \rightarrow \infty \), converges to a Gaussian distribution, \( \mathcal{N}(0, \gamma) \), where, 0 is the mean and, \( \gamma \) is defined as \( Cov(\tilde{X})^{(13)} \). Where, \( \tilde{X}_i \) are identical and independent copies of \( \tilde{X} \).

If we apply this theorem to our case, now we define the random vectors as,

\[ \tilde{X}_i = (d^c_i), \quad (3.10) \]

Where, \( c \) stands for whether Alice’s or Bob’s outcomes, i.e. \( a \) or \( b \). Then,

\[ \bar{S}_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N}(\tilde{X}_i - \langle \tilde{X}_i \rangle) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N}(d^c_i - \langle d^c_i \rangle) = \sum_{i=1}^{N} d^c_i = \bar{I} \quad (3.11) \]

So, the intensity fluctuations converge to a Gaussian distribution, with the mean value as \( 0 \), and a covariance matrix as,

\[ \gamma^{XY}_{cc'} = \frac{1}{N} \sum_{i,j=1}^{N}(d^c_i - \langle d^c_i \rangle)(d^{c'i}_j - \langle d^{c'i}_j \rangle) \]

\[ = \frac{1}{N} \sum_{i,j=1}^{N}(\sqrt{N} \overline{d^c_i})(\sqrt{N} \overline{d^{c'i}_j})) \]

\[ = \langle \bar{I} \overline{I^{c'i}} \rangle \quad (3.12) \]

Also, by assuming that, the particles pairs are not correlated, we can simply conclude,

\[ \gamma^{XY}_{cc'} = \langle \overline{d^c_i d^{c'i}_j} \rangle, \quad (3.13) \]

This, (3.13), can be expressed in terms of microscopic correlations, i.e. \( P(a, b) \). So, in the thermodynamic limit, \( P(a, b) \) are mapped to Gaussian distributions.

All that remains is to show that, there is a \textbf{global} probability density which, these Gaussian marginals could be obtainable from that.
Necessary and sufficient conditions for $P(a, b)$ to be macroscopically local

Macroscopic locality requires the existence of some global distribution, however, \textit{as a consequence}, it must be Gaussian.

So, it leads to the existence of a semi-definite positive global covariant matrix $\Gamma$, which is defined as,

$$\Gamma \equiv \langle \overline{I^C} \overline{I^{C'}} \rangle, \quad (3.14)$$

Where, $C$ and $C'$ belong to any possible measurement outcomes of Alice or Bob.

The positivity could be shown as,

$$\tilde{m}^T \Gamma \tilde{m} = \sum_{C,C'} m_C m_{C'} \langle \overline{I^C} \overline{I^{C'}} \rangle = \langle (\sum_c m_c \overline{I^C})^2 \rangle, \quad (3.15)$$

Where, $\tilde{m}$ is an arbitrary vector. Also, we have used the fact that, there is a global Gaussian distribution for all possible intensity fluctuations.

Now, $\Gamma$, can be written as,

$$\Gamma = \begin{pmatrix} Q & P \\ P^T & R \end{pmatrix} \geq 0, \quad (3.16)$$

Where, the entries are,

$$P_{a,b} = P(a, b) - P(a)P(b) \quad (3.17)$$

$$Q_{a,a'} = \delta_{aa'} P(a) - P(a)P(a') \quad (3.18)$$

$$R_{b,b'} = \delta_{bb'} P(b) - P(b)P(b') \quad (3.19)$$

Where, in the two last equations we have assumed that, $a$ and $a'$ belong to the same measurement setting, the same assumption also holds for $b$ and $b'$.

We could see that, our previous $\gamma^{xy}$ are sub - matrices of $\Gamma$. For example,

$$\Gamma_{a,b} = \langle \overline{I^a} \overline{I^b} \rangle$$

$$= \langle \overline{d_1^a d_1^b} \rangle = \langle d_1^a - P(a) \rangle \langle d_1^b - P(b) \rangle$$

$$= \langle d_1^a d_1^b \rangle - 2P(a)P(b) + P(a)P(b)$$
\[ = P(a, b) - P(a)P(b) \quad (3.20) \]

One should pay attention that, not all of the entries of \( \Gamma \) in (3.16) are determined. For example, if \( b \) and \( b' \) do not belong to the same measurement setting, \( R_{b,b'} \) is not determined. But, actually, this is not a dire consequence, because if there exists a global (Gaussian) distribution which the marginals, page 8, could be obtainable from that, there is always a way to complete \( \Gamma \) by some real numbers, in such a way that, \( \Gamma \) be semi–definite positive.

The vice versa also holds, that is, if there are some real numbers in such a way that our matrix is semi–definite positive, there must be a global (Gaussian) distribution with covariant matrix \( \Gamma \).

Hence, we have obtained some necessary and sufficient conditions in order \( P(a, b) \) to be macroscopically local.

3.2 \( Q^1 \Leftrightarrow \) macroscopic locality

In 2007 and 2008, Miguel Navascués et. al\(^{(7),(8)}\), proposed a way to approximate quantum correlations, \( Q \). More precisely, they obtained a series of correlation sets which, asymptotically converge to the set of quantum correlations, i. e. \( Q \).
Let us, illustrate it more. If we have a quantum state, say $|\Psi\rangle$, and some measurement operators, say $O$, which could be the product of an arbitrary number of operators that, lead to the observed distribution; we could construct a matrix called $\gamma$ with the entries,

$$\gamma_{ij} = \langle \Psi | O_i^\dagger O_j | \Psi \rangle = tr(O_i^\dagger O_j \rho_{AB}),$$  \hspace{1cm} (3.21)

Where, $\gamma$ is always non-negative.

Now, we could define a set of $\gamma_1, \gamma_2, \ldots, \gamma_n$ for any quantum behavior $P(a, b)$ according to the number of measurement operators which are multiplied together, i.e. $\gamma_1$ is constructed for one operator, $\gamma_2$ is constructed for the product of two operators and etc, in a particular iterated fashion, e.g. $\gamma_1$ is constructed for the measurements $O = \{O_a^X \otimes 1_B \} \cup \{1_A \otimes O_b^Y\}$, where, $O_a^X$ denotes the Alice’s particular measurement $X$ which, results $a$ as its outcome and, the same holds for Bob; also, the entries are calculated from $\gamma_{ij} = tr(O_i^\dagger O_j \rho_{AB})$,

$$\gamma_1 = \begin{pmatrix} 1 & O_a^X \otimes 1_B & 1_A \otimes O_b^Y \end{pmatrix}$$

Which, leads to,

$$\gamma_1 = \left( \begin{array}{c} 1 \\ P_A^T \\ \bar{P}_A \\ Q \\ \bar{Q} \\ \bar{P}_A \\ \bar{P}_B \\ \bar{R} \end{array} \right)$$

Where, $\bar{P}_A = (P_1(1), \ldots, P_d(1), \ldots, P_s(1), \ldots, P_d(s))$ is a vector probability for Alice which, the lower indexes denote the measurement settings, as previous there are $s$ possibilities, and the detectors in the laboratory which are designated by $1, \ldots, d$, the same holds for Bob. Also,

$$\bar{Q}_{aa'} = \delta_{aa'} P(a)$$  \hspace{1cm} (3.22)

$$\bar{R}_{bb'} = \delta_{bb'} P(b)$$  \hspace{1cm} (3.23)

$$\bar{P}_{ba} = P(a, b)$$  \hspace{1cm} (3.24)
Where, in the first two equations we have assumed that, \(a\) and \(a'\) belong to the same measurement setting, the same assumption also holds for \(b\) and \(b'\).

As before, the matrix is not completely determined, for example, if \(a\) and \(a'\) do not belong to the same measurement setting, \(\hat{Q}_{aa'}\) is not determined.

\(\gamma_2\) is constructed for the product of two measurement operators, i.e. \(O = \{O^X_a \otimes 1_B\} \cup \{1_A \otimes O^Y_b\} \cup \{O^X_a O^X_{a'} \otimes 1_B\} \cup \{1_A \otimes O^Y_b O^Y_{b'}\} \cup \{O^X_a \otimes O^Y_b\},\)

\[
\gamma_2 = \left(\begin{array}{cc}
\gamma_1 & \\
\end{array}\right)
\]

One could do the same procedure for defining the others \(\gamma_n\), where \(n \in N\).

We know that in quantum physics, all \(\gamma_n\) are non-negative. This fact suggests an algorithm to investigate whether a given microscopic correlation is a quantum one or not. For doing this task, we can define a hierarchy including an infinite examinations, by exploiting these \(\gamma\) and usage of the SDP, i.e. semi-definite programming methods, in other words, correlations of each level, \(Q^1, Q^2, Q^3, \ldots\) are enforced by requiring \(\gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_3 \geq 0, \ldots\).

Actually, we ask, are there some real numbers in order to complete \(\gamma_n\) in such a way that, \(\gamma_n\) be non-negative?

\[
\begin{array}{cccccccc}
\gamma_1 \geq 0 & \text{Yes} & \gamma_2 \geq 0 & \text{Yes} & \gamma_3 \geq 0 & \text{Yes} & \ldots & \gamma_\infty \geq 0 & \text{Yes} \\
\text{No} & \text{No} & \text{No} & \text{No} & \text{No} & \text{No} & \text{No} & \text{No}
\end{array}
\]

As is shown in reference (7), this hierarchy converges asymptotically to quantum correlations, \(Q\), and non-quantum correlations necessarily fail in one of the above steps.
A surprising thing, which was proved one year after the introduction of the above hierarchy, is the fact that $Q^1$, the first level of the above hierarchy, has some physical meaning \(^{(6)}\). Actually, $Q^1$ is equivalent to the set of correlations which emerge from the two principles of no–signaling condition and macroscopic locality.

We mentioned that, $P(a, b)$ belong to the set $Q^1$, if there exist such a semi–definite positive matrix $\gamma_1$,

\[
\gamma_1 = \begin{pmatrix}
1 & P_A^T & P_B^T \\
\overrightarrow{P}_A & \tilde{Q} & \tilde{P}^T \\
\overrightarrow{P}_B & \tilde{p} & \tilde{R}
\end{pmatrix}
\]

Where, the entries are introduced as before.

For proving the equivalence of $\gamma_1$ to the correlations which are macroscopically local, in other words, to prove the equivalence of $\gamma_1$ to $\Gamma$ which was introduced in the section 3.1, we exploit a lemma called Schur’s theorem, apply it to $\gamma_1$, and as a result we obtain $\Gamma$.

**Schur’s theorem**

Suppose $H$ is a matrix of the form,

\[
H = \begin{pmatrix}
E & F \\
F^T & G
\end{pmatrix}
\]

If $E$ is positive, then $G - F^T E^{-1} F \geq 0$ is a necessary and sufficient condition for the non–negativity of $H$. The proof is given in the reference \(^{(9)}\).

By comparison of $H$ and $\gamma_1$, we recognize that, $H$ is $\gamma_1 \geq 0$, $E$ is 1, and, $F$ is $(\overrightarrow{P}_A, \overrightarrow{P}_B)$, and, $G$ is $(\tilde{Q} \ \tilde{P}^T)$. $E$ and $H$ are positive, so it is possible to use Schur’s theorem, and obtain,

\[
\begin{pmatrix}
\tilde{Q} & \tilde{P}^T \\
\tilde{p} & \tilde{R}
\end{pmatrix} - \begin{pmatrix}
\overrightarrow{P}_A \\
\overrightarrow{P}_B
\end{pmatrix} \cdot \begin{pmatrix}
\overrightarrow{P}_A \\
\overrightarrow{P}_B
\end{pmatrix}^T = \Gamma \geq 0
\]

\[(3.26)\]
Hence,

\[ P(a, b) \text{ satisfies macroscopic locality } \iff P(a, b) \in Q^1 \]

As a consequence, quantum correlations are macroscopically local. (Recall that, \( Q \subset Q^1 \))

In the macroscopic locality context, we can picture,

![Diagram](image)

4. Applications

By considering the macroscopic locality, i. e. \( Q^1 \), and also, a hierarchy of necessary conditions which has been introduced in section 3.2, many well – known restrictions on quantum correlations can be recovered. Here, we obtain the Tsirelson – Landau – Masanes inequality; and also, show that, the maximum value of CHSH inequality in \( Q^1 \) is \( 2\sqrt{2} \).

**Tsirelson – Landau – Masanes inequality**

We assume Alice and Bob can each perform two possible measurements, labeled by \( X = 0, 1 \) and \( Y = 0, 1 \) respectively, and, each measurement yields one of the two possible outcomes 0 or 1. So, there are 16 probabilities to characterize the whole system,
\[ P(a, b|X, Y) = (P(0,0|0,0), ..., P(1,1|1,1)), \quad (4.1) \]

Where, \(a\) and \(b\) denote the outcomes corresponding to Alice’s and Bob’s measurements, respectively. We treat this scenario and, construct the matrix \(\gamma_1\), by specifying the correlation functions, i.e. \(C(X, Y) = \sum_{a,b} abP_{(aX)(bY)}\), and, marginal quantities, i.e. one–point correlators, \(C_X = \sum_a aP_{aX}\) and \(C_Y = \sum_b bP_{bY}\); we obtain,

\[
\gamma_1 = \begin{pmatrix}
1 & C(0) & C(1) & C(0) & C(1) \\
C(0) & 1 & u & C(0,0) & C(0,1) \\
C(1) & u & 1 & C(1,0) & C(1,1) \\
C(0) & C(0,0) & C(0,1) & 1 & v \\
C(1) & C(1,0) & C(1,1) & v & 1
\end{pmatrix},
\]

Where, \(u\) and \(v\) are undetermined. If we assume that, we have a quantum physical situation, we know that, the matrix \(\gamma_1\) is semi–definite positive. Now, we could ask, for what values of parameters \(u\) and \(v\), the \(\gamma_1\) is semi–definite positive. But instead of answering this question, if \(\gamma_1\) is semi–definite positive, the matrix,

\[
\overline{\gamma}_1 = \begin{pmatrix}
1 & u & C(0,0) & C(0,1) \\
u & 1 & C(1,0) & C(1,1) \\
C(0,0) & C(0,1) & 1 & v \\
C(1,0) & C(1,1) & v & 1
\end{pmatrix},
\]

Must be also semi–definite positive (a necessary condition). Now, this is the same problem which, Lawrence Landau has solved it and, obtained the inequality,
\[ |\arcsin(C_{13}) + \arcsin(C_{14}) + \arcsin(C_{23}) - \arcsin(C_{24})| \leq \pi, \quad (4.2) \]

Where,

\[
\begin{align*}
C_{13} &= C(0,0) \\
C_{14} &= C(0,1) \\
C_{23} &= C(1,0) \\
C_{24} &= C(1,1)
\end{align*}
\]

Which is the well-known Tsirelson–Landau–Masanes inequality. (Tsirelson\(^{(11)}\) and Masanes\(^{(12)}\) have obtained it, in a different way.)

This condition, discarding the one-point correlators, is necessary for the quantum theory. But, in a general case, where our microscopic correlations are macroscopically local, in other words, \((a, b) \in Q^1\), we must preserve all the elements of the matrix, in order to be able to construct the full probability distribution.

**Tsirelson bound**

One could apply the same procedure which, leads to (4.2), to the matrix \(\gamma_1\), and find an inequality which, must be satisfied, in order \(\gamma_1\) to be semi-definite positive\(^{(8)}\),

\[ |\arcsin(D_{13}) + \arcsin(D_{14}) + \arcsin(D_{23}) - \arcsin(D_{24})| \leq \pi, \quad (4.3) \]

Where,

\[ D_{ij} = \frac{C_{ij} - C_i C_j}{\sqrt{(1-C_i^2)(1-C_j^2)}} \quad (4.4) \]

Also, \(C_i\) and \(C_j\) are the one-point correlators in the matrix.
By exploiting the semi–definite programming methods, we can obtain the maximum violation of the CHSH inequality in $Q$. (Similarly for $Q^1$, because the equations (4.3) and (4.2) have a same form.)

CHSH inequality is a linear combination of probabilities,

$$C_{13} + C_{14} + C_{23} - C_{24} \leq 2$$  \hspace{1cm} (4.5)

We can see that, the above equation is the linear combination of the entries of $\overline{\gamma}_1$.

For obtaining the upper bound of the inequality (4.5), for example in the case of $Q$, we must maximize the inequality, and also, take the care of the constraint $\overline{\gamma}_1 \geq 0$. This optimization is the subject of the semi–definite programming, a special case of convex optimization$^{(14)}$. Reference (15) has discussed the convex optimization in a great detail.

In brief, the semi–definite programming, concerns maximizing such an expression,

$$tr(C\gamma),$$

Subjecting to,

$$\gamma \geq 0,$$

For a given $C$.

We would like to maximize (4.5), so we can simply construct $C$ in such a way that, $tr(C\overline{\gamma}_1)$ leads to (4.5),

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

One could see, $C_{13} + C_{14} + C_{23} - C_{24} = \frac{1}{2} tr(C\overline{\gamma}_1)$. 

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Stephanie Wehner has solved this optimization problem in reference (14), and the result is, $2\sqrt{2}$, the famous Tsirelson bound.

5. Recapitulation

We introduced the concept of macroscopic locality and showed if, one considers it as a fundamental principle, how it discards some un–physical microscopic correlations which are consistent with the no–signaling principle. Also, we argued that quantum correlations are macroscopically local but also, the two sets $Q$ and $Q^1$ are not completely equal; and, we recovered the Tsirelson bound for $Q$ ($Q^1$).
References


