Quantum correlations and generalized probabilistic theories: an introduction
Lecture 1, 16.04.2014

These are unofficial, handwritten lecture notes. Probably with lots of mistakes.
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This course: MVSpec (Advanced Lecture on Special Topic, 4 Credit Points. Exercise towards end of semester.)

1. Overview and perspective on this course

Studying the "Foundations of Quantum Theory" can mean different things:

- Provide an adequate interpretation

- Explore nonclassical phenomena (Bell inequality violations etc.)

- Determine principles from which the quantum formalism may be derived.

Comparison to special relativity:

- Lorentz transformations
- Relativity Principle of Light Postulate
- Some physical principles
- Math. formalism of quantum theory
Suggestive Comparison:

THEORIES OF (SEMI-RICHTHOFEN) GEOMETRY

- standard Euclidean geometry
- Minkowski Space (locally flat)

GENERALIZED PROBABILISTIC THEORIES

- classical probabilistic theory
- quantum theory (Q)

Why GPTs?
- understand what makes QT special, and why
- give consistent alternatives

Weibel 1988: "Testy Quantum Mechanics"
- introduce nonlocal correctness to QT
- showed that Weibel's approach allows for superluminal signalling

- new experiments to test QT
- implications for quantum gravity
- fully mathematically rigorous, minimal phys. assumptions
- application: device-independent cryptography

Fundamental understanding of quantum formalism
- argue counterfactually about relation of QT and spacetime
2. General behaviors and the Bell–CHSH inequality

General setup:

The two black boxes are measurement devices: the first specifies what measurement to perform, and the second gives the measurement result.

Useful convention: \(a, b, c \in \{0, 1\}\), \(x, y \in \{+1, -1\}\).

Examples:

- **Quantum theory**: the preparation device prepares an entangled state of two spin-\(\frac{1}{2}\) particles, and distributes them to Alice and Bob.
  
  Alice can choose to measure the spin in \(x\)-direction (\(a = 0\)) or in \(y\)-direction (\(a = 1\)).
  
  She obtains either "spin-up" (\(x = +1\)) or "spin-down" (\(x = -1\)).
  
  Same for Bob.

- **Classical**: Bertrand's socks are randomly put into two parcels and sent to Alice and Bob.
  
  0-measured: spits out "-1" if green, "+1" if red.
  
  1-measured: spits out "-1" if left foot, "+1" if right foot.

For a given choice of preparation & measurement devices, set

\[ C_{a,b} := E(x - y | a, b). \]

Bertrand's socks will have

\[ P(x = \pm 1, y = \pm 1 | a = 0, b = 0) = 0, \]
\[
P(x = \pm 1, \ y = \pm 1 | a = 0, \ b = 0) = \frac{1}{2} \Rightarrow x \cdot y = -1 \text{ always} = \Rightarrow E(x \cdot y | 0, 0) = -1 = C_{00}
\]

Similar calculation \( \Rightarrow C_{11} = -1, \ C_{01} = C_{10} = +1 \).

Claim: For classical correlations, we always have

\[
|C_{00} + C_{01} + C_{10} - C_{11}| \leq 2
\]

(Bell-CHSH inequality).

**Def:** A \((2,2,2)-\) behavior is a function

\[
P: \{(x, y, a, b) \} \rightarrow P(x, y | a, b) \in [0,1],
\]

\(x, y \in \{-1, +1\}, \ a, b \in \{0,1\} \)

with \( \sum_{x,y} P(x, y | a, b) = 1 \) \( \forall a, b \).

Which of these \(P\)’s can we realize within classical physics / quantum physics under locality assumptions?

Classical prob theory: We have a sample space \(\Omega\), where \(\Omega = \{0, 1\} \times \{0, 1\} \times \Lambda\), where \(\Lambda\) describes the preparation and all other variables ("elements of reality") that are potentially relevant for the experimental outcomes.

Example: \(a = -1 \) if \(\xrightarrow{\text{left}}\) and \(a = +1 \) if \(\xrightarrow{\text{right}}\).
For simplicity, let's assume here that $\Lambda$ is finite/discrete (otherwise some conclusions, but a bit of measure theory). We also have a joint probability dist. $\mathbf{Q} = \mathbf{Q}(a, b, \lambda)$ on $\Omega$.

Random variables $x, y$ are functions of sample space elements:

\[ x = f(a, b, \lambda), \quad y = g(a, b, \lambda). \]

Space-like separation is formalized as $x = f(a, \lambda)$, $y = g(b, \lambda)$.

"Free choice": $Q(a, b, \lambda) = Q_A(a) \cdot Q_B(b) \cdot Q_{\Lambda}(\lambda)$.

(Local inputs uncorrelated with hidden variables and themselves)

**Def.** A $(2, 2, 2)$-behavior $P$ is classical if there exists $\Lambda$ and a joint distribution $\mathbf{Q}$ on $\{0, 1\} \times \{0, 1\} \times \Lambda$ such that $P(x, y | a, b)$ is equal to the conditional probability $Q(x, y | a, b)$.

**Proposition.** If $P$ is classical then $C := C_{00} + C_{10} + C_{01} - C_{11}$ satisfies $|C| \leq 2$.

**Proof.**

\[ C_{a,b} = E(x \cdot y | a, b), \quad x \cdot y = f(a, \lambda) \cdot g(b, \lambda) \]

\[ \implies \mathcal{E}(\lambda) := \mathbf{Q}(0, a) \cdot \mathbf{Q}(0, \lambda) \cdot \mathbf{Q}(1, a) \cdot \mathbf{Q}(1, \lambda) + \mathbf{Q}(0, a) \cdot \mathbf{Q}(1, \lambda) + \mathbf{Q}(1, a) \cdot \mathbf{Q}(0, \lambda) - \mathbf{Q}(1, a) \cdot \mathbf{Q}(1, \lambda) \]

\[ = \begin{cases} \pm 1 & \text{if either one of them is 0, the other is } \pm 2 \\ \pm 1 & \text{if both are } \pm 2 \end{cases} \]

\[ = \pm 2 \in [-2, +2] \]
Now \( C_{a,b} = \mathbb{E}(x, y | a, b) = \sum_{x, y} (x, y) \cdot Q(x, y | a, b) \)
\[
= \sum_{x, y} (x, y) \frac{Q(x, y, a, b)}{Q(a, b)} = \sum_{x, y} \left( x, y \right) \frac{\sum_{\lambda \in \Lambda} Q(x, y, a, b, \lambda)}{Q(a, b)}
\]
\[
= \sum_{x, y} (x, y) Q(a, b)^{-1} \sum_{\lambda \in \Lambda} \delta_{x, f(a, \lambda)} \delta_{y, g(b, \lambda)} Q(\lambda).
\]
\[
C = \sum_{\lambda \in \Lambda} l(\lambda) Q_{\lambda}(\lambda) = \mathbb{E}_{Q(\lambda)} l(\lambda) \in [-2, +2].
\]

**Def.** A \((2, 2, 2)\)-behavior \( P \) is quantum if there exist Hilbert spaces \( \mathcal{H}_A, \mathcal{H}_B \) and a pure state \( |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) as well as orthogonal projectors \( \Pi_a^x \) on \( \mathcal{H}_A \), \( \Pi_b^y \) on \( \mathcal{H}_B \) with
\[
\Pi_a^{x=-1} + \Pi_a^{x=+1} = \Pi_A, \quad \Pi_b^{y=-1} + \Pi_b^{y=+1} = \Pi_B, \quad \text{and}
\]
\[
P(x, y | a, b) = \langle \psi | \Pi_a^x \otimes \Pi_b^y | \psi \rangle.
\]

**Observation**: There are quantum behaviors that violate the Bell–CHSH inequality, for example.
147 = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \text{ on } \mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2

(for example, spin-\frac{1}{2} particle)

\Pi_{a=0} \text{ project onto the corresponding eigenspaces of } G_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}

\Pi_{a=1} \text{ project onto } \sigma_x \text{ of } G_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}

\Pi_{b=0} \quad \quad \quad \frac{1}{\sqrt{2}} (G_y - G_x)

\Pi_{b=1} \quad \quad \quad \frac{1}{\sqrt{2}} (G_x + G_y)

\text{achieves } C := C_{00} + C_{01} + C_{10} - C_{11} = 2\sqrt{2} = 2.82...

\text{Violation of Bell inequality:}

\text{Non-local correlations; Bell's theorem; more on this next time.}

\text{Tsitelison bound: } 2\sqrt{2} \text{ is the natural } CHSH \text{ value}

\text{for any quantum behavior.} \quad \text{(Proof: later)}

Can A & B use 147 to send information?

\text{Def.: A behavior } P \text{ satisfies the no-signalling condition if}

\text{a) } P(x|a) \text{ does not depend on } b \text{ (no-sign from Bob to Alice),}

\text{and}

\text{b) } P(y|b) \text{ does not depend on } a \text{ (no-sign from Alice to Bob)}

\sum_{y} P(y|x,a,b) = \sum_{y} P(y|x,g|a,b') \forall a, b, x, y

\sum_{y} \frac{P(x,y|a,b)}{P(x|a,b)} = \sum_{y} \frac{P(x,y|a,b')}{P(x|a,b')} \forall x a b
\[ \sum_{x} P(x|y|a,b) = \sum_{x} \underbrace{P(x|y|a',b)}_{p(y|a,b)} V y, b. \]

Claim: Quantum behaviors are non-signalling.

Proof (of first part only):

\[ \sum_{y} P(x|y|a,b) = \sum_{y} \langle 4| \Pi_{a}^{x} \otimes \Pi_{b}^{y} | 4 \rangle = \]

\[ = \langle 4| \Pi_{a}^{x} \otimes (\sum_{y} \Pi_{b}^{y}) | 4 \rangle = \]

\[ = \langle 4| \Pi_{a}^{x} \otimes 1 | 4 \rangle \]

does not depend on b, i.e. \( = \sum_{y} P(x|y|a,b) \).

Second equation analogous.

\( \square \)

Classical behaviors are non-signalling, too (they can be achieved by quantum states).

Popescu + Rohrlich 1993 asked: is \( C=2\sqrt{2} \) the "strongest" correlation that fits into relativity?

Surprising answer: NO! The "PR box" (behavior) \( \text{PR} \) is a counterexample:

\[ \forall (a,b) \in \{(0,0), (0,1), (1,0)\} : \]

\[ \text{PR}(+1,+1|a,b) = \text{PR}(-1,-1|a,b) = \frac{1}{2} \] (perfect correlation), and

\[ \text{PR}(+1,-1|1,1) = \text{PR}(-1,+1|1,1) = \frac{1}{2} \] (perfect anticorrelation).
It turns out that $\text{PR}$ satisfies the no-signalling conditions, and

$$C = C_{00} + C_{01} + C_{10} - C_{11} = 4 > 2\sqrt{2}.$$