7.3. Definition of state spaces (continued)

Def.: State spaces \((A, A_+, \Omega_A)\) and \((B, B_+, \Omega_B)\) are equivalent if there is an invertible linear map \(L: A \to B\) such that \(B_+ = L(A_+)\) and \(\Omega_B = \Omega_A \circ L^{-1}\).

\((\dagger)\) is equivalent to \(L(\Omega_A) = \Omega_B\): if \(\omega_B = L(\omega_A)\) then \(\omega_B(\omega_B) = (\Omega_A \circ L^{-1})(L(\omega_A)) = \Omega_A(\omega_A)\).

Example: quantum bit.

\[A = \{ M \in \mathbb{C}^{2 \times 2} \mid M = M^+ \}\]
\[A_+ = \{ M \in A \mid M \text{ positive-semidefinite} \} \quad (M \geq 0)\]
\[\Omega_A = \{ \sigma \in A \mid \sigma \geq 0, \quad tr \sigma = 1 \}\]
\[\Omega_A(M) = tr(M)\]

\[B = \mathbb{R}^4\]
\[\Omega_B = \{ (\tau^T) \mid \tau \in \mathbb{R}^3, \quad |\tau| \leq 1 \}\]
\[B_+ = \mathbb{R}_+ \cdot \Omega_B\]
\[\Omega_A(x) = \omega_A(x) = x_0\]
These state spaces are equivalent descriptions of
the quantum bit:

\[ L : \mathcal{B} \to \mathcal{A} \]

\[
\begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\mapsto
\frac{1}{2}
\begin{pmatrix}
  x_0 + x_3 \\
  x_1 + x_2 \\
  x_1 - i x_2 \\
  x_0 - x_3
\end{pmatrix}
\]

\[
= \frac{1}{2}
\begin{pmatrix}
  x_0 + x_3 & x_1 - i x_2 \\
  x_1 + x_2 & x_0 - x_3
\end{pmatrix}
\tag{2}
\]

Clearly, the result is self-adjoint, and \( L \) is linear and
invertible.

\( x \in \mathcal{B} \Rightarrow x = \begin{pmatrix} 1 \\ x \end{pmatrix}, \quad x \in \mathbb{R}^3, \quad |x| \leq 1 \)

\( x_0 = 1 \Rightarrow L(x) = 1 \)

Eigenvalues of \( L(x) \): \( \lambda_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{x_1^2 + x_2^2 + x_3^2} \right) \)

\[
= \frac{1}{2} \left( 1 \pm |x| \right) \geq 0
\]

\( \Rightarrow L(x) \in \mathcal{P}_A \).

Conversely, every \( s \in \mathcal{P}_A \) can be written in the form (2)
with \( x_0 = 1 \), and \( \lambda_{1,2} > 0 \Rightarrow |x| = |(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})| \leq 1 \).

State spaces are equivalent if and only if they give
exactly the same (statistical) physical predictions.

**Proposition**: Every state space \((A, A_t, U_t)\) is equivalent to
some \((B, B_t, U_B)\) such that all \( W_B \in \mathcal{P}_B \)
have numbers between 0 and 1 as components (probabilities of fiducial measurement outcomes).
"Proof" by picture:

\[ B = \mathbb{R}^{\text{dim} A} \]

\[ L(\mathbb{R}^n) = \mathbb{R}^n \]

Works since \( \mathbb{R}^n \) is compact. \( \square \)

So we are back where we started in the first place!

7.4. Transformations

\[ T \text{ maps a system of type } A \]
\[ \text{ to a system of type } B. \]

Suppose \( P \) prepares \( \varphi_A \) with prob. \( \lambda \), and \( \varphi_A' \) with prob. \( (1-\lambda) \)

Description 1: We have \( \varphi_B = \lambda \varphi_A + (1-\lambda) \varphi_A' \),
which is mapped to \( \varphi_B = T(\varphi_A) \).

Description 2: With prob. \( \lambda \), we have \( \varphi_A \) and obtain \( T(\varphi_A) \);
with prob. \( (1-\lambda) \), we have \( \varphi_A' \) and obtain \( T(\varphi_A') \)
\( \Rightarrow \) we obtain \( \varphi_B = \lambda T(\varphi_A) + (1-\lambda) T(\varphi_A') \)

Both descriptions valid
\[ T(\lambda \varphi_A + (1-\lambda) \varphi_A') = \lambda T(\varphi_A) + (1-\lambda) T(\varphi_A) \]

Def. (Transformation)

Let \( (A, A_+, A_-) \) and \( (B, B_+, B_-) \) be state spaces. A linear map \( L : A \rightarrow B \) is called
- positive, if \( L(A_+) \subseteq B_+ \)
- normalization-nonincreasing if \( u_B(L(w_A)) \leq u_B(w_A) \) \( \forall w_A \in A_+ \).
A is called a transformation if it is positive and normalization non-increasing.

A transformation $T$ is called reversible if it is invertible and if $T^{-1}$ is a transformation, too. ($\Rightarrow$ $T$ is normalization-preserving: $u_B = u_A$).

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

$\Rightarrow$ A and $B$ are equivalent if and only if there exists a reversible transformation between them.

The reversible transformations $T : A \to A$ form a group that we denote $G_A$. It is a compact (Lie) group. From now on consider only normalization-preserving maps. They satisfy $T(\mathbb{R}) \subseteq \mathbb{R}$.

Example: $G_{bit}$.

T acts linearly on $\mathbb{R}^3$, preserves $\text{span} (\mathcal{N}_A)$.

$\Rightarrow$ T acts affine-linearly on the plane $\text{span} (\mathcal{N}_A)$.

Look only at this plane.

T is a rotation by $k \cdot 90^\circ$ and/or a reflection at center.

$T^{-1} = G_{\max} = D_4$, the dihedral group (of the square).
Without symmetries, $G_A$ is trivial:

$\nabla_A \Rightarrow G^\text{max}_A = \{ \mathbb{1} \}$.

**Lemma:** Reversible transformations map pure states to pure states.

**Proof:** Suppose $T: A \rightarrow A$ is reversible and $\rho = T(\rho)$.

$w, \rho \in \mathcal{S}_A$.

Suppose $\rho$ is mixed, i.e.

$\rho = \lambda \rho_1 + (1-\lambda) \rho_2, \quad \lambda \neq \rho_2, \quad 0 < \lambda < 1$

$w_i := T^* \rho_i \Rightarrow w = T^* \rho = \lambda w_1 + (1-\lambda) w_2, \quad w \neq w_i, \quad 0 < \lambda < 1$

$\Rightarrow w$ mixed.

**Example:** Classical prob. theory.

$\Omega_B = \{ \left( \begin{array}{l} p_1 \\ \vdots \\ p_n \end{array} \right) \in \mathbb{R}^n \mid p_1 + \ldots + p_n = 1, \quad p_i > 0 \}$

$T: B \rightarrow B$ tranformation $\Rightarrow T(\rho_B) \in \mathcal{S}_B$

$\Rightarrow$ columns of $T$, $T(\frac{1}{\sqrt{2}}), \ldots, \frac{1}{\sqrt{2}})$ etc., must be probability distributions $\Rightarrow T$ is a stochastic matrix

(a classical channel)

$T$ is a reversible transformation $\Rightarrow T$ is a stochastic matrix, and maps the set of pure states $\{(\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}), \ldots \}$ to itself $\Rightarrow T$ is a permutation matrix

$T \left( \begin{array}{l} p_1 \\ \vdots \\ p_n \end{array} \right) = \left( \begin{array}{l} P_{11} (p_1) \\ \vdots \\ P_{nn} (p_n) \end{array} \right)$, $T \in S_n$

$\Rightarrow G^\text{max}_B = S_n$, the permutation group

**Example:** Quantum theory.

$\Omega_C = \{ \left| \phi \right\rangle \in \mathbb{C}^n \mid \langle \phi | \phi \rangle = 1 \}$

$C = \{ A \in \mathbb{C}^{n \times n} \mid A^\dagger = A \}$

vector space
There are many transformations $T : C \rightarrow C$
(more details/examples later when we talk about composite systems), for example open system (Lindblad) evolutions $T(s) = s(t)$
for solution of $s(0) = s$

$s = -\frac{i}{h} [H, s] +$
$+ \sum_{nm} \hbar \Omega(m)(L_n s L^+_m$
$- \frac{1}{2} (s L^+_m L_n + L^+_m L_n s))$

What are the reversible transformations?

Wigner's Theorem (in our vocabulary):

$G^\text{max}_C = \{ T | T(s) = U s U^{-1}, \text{ with either} \}
\begin{cases} 
U \text{ linear and unitary,} \\
\text{or } U \text{ and linear and antiunitary} 
\end{cases}
\begin{align*}
= PU A(n) & \quad U(ax + by) = a U(x) + b U(y) \\
\text{the projective unitary antiunitary group.} & 
\end{align*}

On single (isolated) systems, antiunitary maps are perfectly fine. But if applied to parts of composite systems, they map states to non-states; i.e. $T$ positive, but $T \otimes U$ not positive.

Ex.: $T(s) = s^T$ is of the form USU$^{-1}$ with U antiunitary.
The $T \otimes U (\phi_+ \otimes \phi_+)$ has negative eigenvalues.

Proof: Exercise, or Nielsen-Chuang.
Def: A state space \((A, A', U_0)\) together with a closed subgroup \(G_1 \subset G_{\text{max}}\) is called a dynamical state space.

Interpretation: The \(G_1\) are those rev tranf. that can actually be physically implemented.

In quantum theory, \(G_1 = \{ T | \ T(8) = U8U^+, \text{ Unitary} \}\)

the projective unitary group.

If \(s\) is pure, \(s|\psi\rangle = |\psi\rangle\Rightarrow U8U^+ = (U|\psi\rangle)(U|\psi\rangle)^+

We can look at \(|\psi\rangle \rightarrow U|\psi\rangle\) separately.

Linearity in the state vector \(|\psi\rangle\) is an accidental math. consequence of reversibility, not a fundamental property of nature! Linearly in density matrix \(\rho\).

For a qubit \((n=2)\), unitaries act as rotations, and anti-unitaries as reflections.

In this case, \(G_1 \cong SU(2)\), \(G_{\text{max}} \cong SO(3)\).

Interesting: QT allows for continuous reversible time evolution, but CPT does not!