Exercises 6

(to hand in: December 16, 2014)

Problem 23 (The classical Ising model and typical subspaces): (15 points)

In this exercise, we explore the formal similarities in the description of simple statistical systems and the information theoretic view on a string of length n with just two different symbols.
We consider the classical non-interacting Ising model in one dimension, where we have a chain of n "spins" σ_i, and each one can have the values $+1$ (up) or -1 (down). The configuration space of the model in the canonical ensemble is thus $\Omega = \{-1, 1\}^n$. The energy function for a spin chain $\sigma = (\sigma_1 \ldots \sigma_n)$ is given by

$$H = -h \sum_{i=1}^{n} \sigma_i,$$

where $h \in \mathbb{R}$ is some external parameter. The probability for a specific configuration is given by the Boltzmann distribution

$$P(\sigma) = e^{-\beta H(\sigma)}/Z \qquad \text{with} \qquad Z = \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)}.$$

(i) What are the probabilities for a single spin to be in the up or down state? Write down the Boltzmann entropy s for a single spin (i.e. a spin chain of length one). Argue that the Boltzmann entropy of a chain of size n is $n \cdot s$. (Don't calculate, use a general property of the entropy.) We thus call s the entropy density of the model.

(ii) What is the expectation value $\langle H \rangle$ for the energy function H for a single spin and in the case of n spins?

We define the set of ϵ-typical states $T_\epsilon \subset \Omega$ on a chain of length n as

$$T_\epsilon = \left\{ \sigma \in \Omega \,\middle|\, s - \epsilon < -\frac{1}{n} \log P(\sigma) < s + \epsilon \right\}.$$

(iii) Prove now that this set contains exactly the states of the microcanonical ensemble, i.e.

$$T_\epsilon = \{ \sigma \in \Omega \mid \langle H \rangle - \epsilon\, n/\beta < H(\sigma) < \langle H \rangle + \epsilon\, n/\beta \}.$$

Hints: Start by rewriting the definition of T_ϵ as a restriction on the number of up states in the configuration σ, by the the expectation value of the number of up spins. Since the energy obviously depends on this number of up states, continue to transform this into a requirement on the energy of the configuration. You will need the explicit form of s written down in (i).

(iv) Explain in words (without formal details) why this can be interpreted as an instance of "equivalence of ensembles".

Problem 24 (Catalysis in small dimensions): (15 points)

We introduced the notion of catalysis in the lecture to enable state transitions under noisy operations of states that were incomparable before.

(i) Explain why catalysis is not of much interest if we have two probability vectors which are both just of dimension two.

As well in dimension three catalysis can't enable transitions that are forbidden without a catalyst. To show this, first prove the following.

(ii) Let p and q be probability vectors of size three and assume that for some catalyst c we have
$$p \otimes c \xrightarrow{\text{noisy}} q \otimes c.$$
Show that $p_1^\downarrow \geq q_1^\downarrow$ and $p_3^\downarrow \leq q_3^\downarrow$ must hold. (Hint: You can assume p, q and c to be decreasingly ordered.)

(iii) Now suppose that p and q are incomparable, i.e. $p \not\succ q$ and $q \not\succ p$. Use the above result to show that no c exists such that $p \otimes c$ and $q \otimes c$ are comparable.

(iv) Choose states $p = (0.5, 0.25, 0.25, 0)$, $q = (0.8, 0.2)$ and a catalyst $c = (0.6, 0.4)$. Show that $p \nrightarrow q$ by noisy operations but $p \otimes c \xrightarrow{\text{noisy}} q \otimes c$ is possible.

Problem 25 (Bonus: Data compression and Shannon's Theorem): (+10 points)

We learned in the lecture and in the tutorial that the Shannon entropy quantifies how many bits are needed to encode the output of some information source. Choose the programming language of your choice and fill a few text documents with random zeros and ones. For the different documents choose different probabilities for the zero to occur. If you write the same amount of symbols into each file they should all have the same size. Now take a file compression tool and compress all these files. What do you expect from Shannons theorem for the sizes of the compressed files depending on the probability for the zero to occur? What do you find?