Exercises 7
(to hand in: January 7, 2015 in the lecture)

In the following, we say that a state \( \rho \) is diagonal on a quantum system \( S \) with Hamiltonian \( H_S \) if and only if \( [\rho, H_S] = 0 \). This is equivalent to the fact that there is some (energy) eigenbasis of \( H_S \) such that the matrix representation of \( \rho \) in this basis is a diagonal matrix.

The Gibbs state at inverse temperature \( \beta \) is \( \gamma_\beta := \exp(-\beta H_S)/Z \), where \( Z \in \mathbb{R} \) is such that \( \text{tr}(\gamma_\beta) = 1 \).

Problem 26 (Gibbs states are completely passive): (8 points)
In the lecture, we have learned that a state \( \rho_S \) is passive (with respect to some Hamiltonian \( H_S \)) if and only if \( \rho_S = \text{diag}(p_1, \ldots, p_n) \) in some eigenbasis of \( H_S \), and \( E_i > E_j \Rightarrow p_i \leq p_j \). A state \( \rho_n \) is called completely passive if \( \rho_n \otimes \mathbb{I} \) is passive for all \( n \). (In this statement, the Hamiltonian on the system \( S^\otimes n \) is taken to be \( H_S \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} + \mathbb{I} \otimes H_S \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes H_S \).) Show that all Gibbs states \( \gamma_\beta \) (for all inverse temperatures \( \beta \geq 0 \)) are passive and completely passive.

Problem 27 (Passive and completely passive states in small dimensions): (8 points)

(i) Argue that every density matrix of full rank can be seen as Gibbs state if you choose the Hamiltonian correctly. Or stated differently, every density matrix can be written as \( \rho = e^{-\beta H}/Z \) if \( H \) and \( \beta \geq 0 \) are chosen appropriately. (You don’t have to be totally mathematically rigorous here.)

(ii) Show that for a fixed diagonal Hamiltonian \( H \) of size two, every diagonal density matrix of full rank is a Gibbs state for some appropriate inverse temperature \( \beta \geq 0 \).

Use Exercise 26 to conclude that all passive states in two dimensions are completely passive.

(iii) Find an example in three dimensions of a state (and a corresponding Hamiltonian) that is passive but not completely passive.

Problem 28 (General properties of thermal operations): (6 points)
Consider thermal operations \( \Phi : S \to S \), where \( S \) is a finite-dimensional quantum system.

(i) Show that all thermal operations leave the Gibbs state of the system invariant, i.e. \( \Phi(\gamma_S) = \gamma_S \).

(ii) Show that the image of any diagonal state under thermal operations is diagonal. Thus, no superpositions of energy eigenstates can be created by thermal operations.
Consider a two-dimensional quantum system $S$ and a two-dimensional heat bath $B$ with respective Hamiltonians

$$H_S = \begin{pmatrix} 0 & 0 \\ 0 & \Delta E \end{pmatrix}, \quad H_B = \begin{pmatrix} E & 0 \\ 0 & E + \Delta E \end{pmatrix}.$$ 

Let $\beta \geq 0$ be some arbitrary but fixed inverse temperature.

(i) Show that the Gibbs states of the system and the bath are identical, i.e. $\gamma_S = \gamma_B$.

(ii) Let the system be in the diagonal state $\rho_S = \text{diag}(p, 1-p)$, where $0 \leq p \leq 1$. Show that under thermal operations the transition $\rho_S \mapsto \gamma_S$ is possible, and give a unitary that (together with a partial trace) achieves this transition.

(iii) What other state transitions are possible with this environment? Give the set of achievable states as a subset of the set of two-component probability vectors. (Of course the states are all matrices, but since they are all diagonal (cf. Exercise 28), we can represent them as probability vectors.)

Hint: Argue that we obtain transitions from $\text{diag}(p, 1-p)$ to $\text{diag}(p', 1-p')$, and that

$$\begin{pmatrix} p' \\ 1-p' \end{pmatrix} = B \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

for some matrix $B$. Use known properties of $B$ to determine all possible $p'$. 
