Recap last lecture:

Free energy as a transition rate:

\[ m_n = \text{largest integer with } A \xrightarrow{\text{on } S_{\text{thermal}}} B, \quad \text{then} \]

\[ \lim_{n \to \infty} \frac{m_n}{n} = \frac{D(S_{A||Y})}{D(S_{B||Y})} = \frac{F_\beta(S_A) - F_\beta(Y)}{F_\beta(S_B) - F_\beta(Y)} \]

where \( F_\beta(g) = k(S(g)) - S(g)/\beta \) (\( = "U - TS" \)).

Reduction to classical case is not exactly possible (in contrast to resource theory of nonunitality):

- not only eigenvalues of state \( S_5 \) important, but also how eigenbasis of \( S_5 \) "is rotated with respect to eigenbasis of \( H_5 \)".

- Many equivalences of resource theory of nonunital do not hold in the resource theory of athermality of.

\[ S \to G \] by Gibbs-preserving map

\[ G \] \( S \to G \) by a thermal operation.

3.5. From the quantum to the classical case?

Continued.

In light of these problems, let's stick to block-diagonal states, i.e. \([S_5, H_5] = 0 \rightarrow \text{everything becomes classical} \).
Classical system: $A = (P_A, H_A)$

$P_A \in \mathbb{R}^n$ probability vector, $H_A = (E_1, \ldots, E_n) \in \mathbb{R}^n$ energies.

A permutation $\pi: \mathbb{R}^n \to \mathbb{R}^n$ is energy-preserving if $\pi[H_A] = H_A$.

**Definition:** A map $D: \mathbb{R}^d_s \to \mathbb{R}^d_s$ is a thermal classical operation if there is a system $E = (f, H_E)$ with

$$(f)_i = \exp(-\beta(H_E)_i)/\beta$$

and an energy-preserving permutation $\pi$ on $SE$ with

$$D(\rho_s) = (\pi[\rho_s \otimes f_E])_{s'}.$$

Let $\rho = (\rho_s, H_s)$.

Write $\rho \twoheadrightarrow \sigma$ if for every $\epsilon > 0$ there is $\sigma_\epsilon$ with

$|\sigma - \sigma_\epsilon| < \epsilon$ and a thermal operation $D_\epsilon$ with

$D_\epsilon(\rho) = \rho_\epsilon$. Analogous def. hold for quantum states.

**Lemma:** Let $\sigma_1, \sigma_2$ be quantum states on $S$.

If $\sigma_1, \sigma_2$ are block-diagonal, then

$\sigma_1 \twoheadrightarrow \sigma_2 \iff 2(\sigma_1) \twoheadrightarrow 2(\sigma_2)$

classically.

where $2(\sigma)$ are the eigenvalues of $\sigma$, ordered in accordance with the ordering of the energy values.

**Proof:** Maybe in homework.
Theorem (Janzing et al. 2000):

Let \( \mathbf{P} = (P_0, N_0) \) and \( \tilde{\mathbf{P}} = (\tilde{P}_0, \tilde{N}_0) \) be classical systems of the same size, i.e. \( P_0, \tilde{P}_0 \in \mathbb{R}^n \). Then \( \mathbf{P} \rightarrow \tilde{\mathbf{P}} \) if and only if there exists a stochastic matrix \( A \) such that \( A P = \tilde{P} \) and \( A g = \tilde{g} \),

where \( g = \exp(-\beta N_0) / Z \) and similarly \( \tilde{g} \) are the Gibbs states.

Important theorem! This will be our main tool in the following.

Proof sketch:

\( \Rightarrow \): In special case \( N_0 = \tilde{N}_0 \), this means that thermal operations preserve the Gibbs state. Prove analogous to quantum case (cf. homework).

\( \Leftarrow \):

Consider special case \( n = 2 \) (i.e. bits).

\( A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \) stochastic matrix, i.e. \( a_{00} + a_{10} = 0, a_{01} + a_{11} = 1, a_{ij} > 0 \). Assume \( A P = \tilde{P} \) and \( A g = \tilde{g} \).
Consider the composite system with Hamiltonian

\[ H + H + H + \cdots + H + H + H \]

(input and output, \( n \) total (2n+1) bits), and a thermal operation

\[ D(p) = \left( \prod_{i} [p \otimes g_{0}^{a_{i}} \otimes g_{0}^{a_{i+1}}] \right)^{2n+2}. \]

\[ \text{Want: this is } \approx \tilde{p}. \]

Construction of the energy-preserving permutation \( \Pi \):

Two binary strings of length \( n \) each are characterized by 4 numbers:

- first bit of the first word, denoted \( j \in \{0, 1\} \)
- number of ones in the first word, denoted \( r \in \{0, n\} \)
- second \( n \) second \( \cdot \) second, denoted \( s \in \{0, n\} \)
- last bit of the last word, denoted \( x \in \{0, 1\} \)

\[ \rightarrow \text{4-tuple } (j, r, s, x). \]

Fix \( r, s \). Then do a permutation of the following kind:

\[ \{(0, r, s, 0)\} \quad \{(1, r, s, 1)\} \]

strings ending with "0" strings ending with "1"
If $n \to \infty$, we can restrict to typical subset.

$$T_n := \{ (j, r, s, x) \mid \frac{r}{n} \approx g_1, \frac{s}{n} \approx g_1 \}$$

approx. uniform distribution on $T_n$

$\Rightarrow$ prob. to be in some subset $\alpha$ relative size of subset.

$\Rightarrow$ Prob (land in $0$-box after permutation)

$= \text{Prob}(\text{start in } M_0 \text{ or } M_1)$

$\leq \text{Prob}(\text{start in green box}) \cdot a_{00}$

$+ \text{Prob}(\text{start in pink box}) \cdot a_{01}$

$\approx P_0 a_{00} + P_1 a_{01} = \frac{1}{n} \sum_{j} a_{0j} d_j = (Ap)_0 = \hat{P}_0$.

Check that we can indeed construct $\tilde{T}_{r,s}$.

Left boxes must fit (tightly) into right boxes:

Need $\# \{ (0, r, s, 0) \} \leq a_{00} \cdot \# \{ (0, r, s, 0) \}$

$+ a_{01} \cdot \# \{ (1, r, s, 0) \}$. (*)

But $\# \{ (0, r, s, 0) \} \approx \hat{g}_0 \cdot \# \{ (0, r, s, 0) \}$ (combinatorics for large $n$)

and $\# \{ (j, r, s, 0) \} \approx \hat{g}_j \cdot \# \{ (r, s, 0) \}$

Then (x) follows from $\hat{g}_0 = a_{00} \hat{g}_0 + a_{01} \hat{g}_1$.

i.e. $\hat{g} = \hat{A} \hat{g}$.
\[ \text{(**) Include's } C \text{ inside } t \text{ bits, } \text{ind } \sim p(1)=p. \]

\[
\begin{align*}
\# \text{ of strings with } s \text{ ones, last bit zero } &= \binom{n-s}{s} \\
\# \text{ of strings with } s \text{ ones, last bit arbitrary } &= \binom{n}{s} \\
\end{align*}
\]

If \( s \approx \alpha n \), this leads to \( (1-p) \) for \( n \to \infty \).

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3.6. \( d \)-majorization and \( d \)-convex curves

**Def.** Let \( p, p', d, d' \in \mathbb{R}^n \) be probability vectors. We say that \( (p, d) \) \( d \)-majorizes \( (p', d') \) if there is a channel (i.e., stochastic matrix) \( B \) with \( Bp = p' \) and \( Bd = d' \). Then we write:

\[ (p, d) \succ (p', d') \]

Special case \( d = d' \), we also write \( p \succ_d p' \).

\[ d = d' = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) : \text{ standard majorization} \]

**Theorem:** For probability vectors \( p, p', d, d' \in \mathbb{R}^n \), the following conditions are equivalent:

(i) \( (p, d) \succ (p', d') \)

(ii) for every convex function \( g \),

\[
\sum_i d_i \cdot g \left( \frac{p_i}{d_i} \right) \geq \sum_i d'_i \cdot g \left( \frac{p_i}{d'_i} \right)
\]
(iii) The $d$-Lorentz curve of $p$ is everywhere on or above the $d'$-Lorentz curve of $p'$.

(iv) Hence $p \rightarrow p'$, if initial and final Hamiltonians are such that $d$ and $d'$ are the corresponding Gibbs states.

Proof: (i) $\Rightarrow$ (iv) is already proven.

Rest maybe in homework.

d-Lorentz curve (also "Gibbs-rescaled Lorentz curve"):

Consider \( \left( \frac{p_i}{d_i} \right)_{i=1,\ldots,n} \) and sort them in decreasing order:

\[ \vec{p} = \Pi(p), \quad \vec{d} = \Pi(d) \quad \text{(same permutation)} \quad \text{such that} \]

\[ \frac{\tilde{p}_1}{d_1} \geq \frac{\tilde{p}_2}{d_2} \geq \frac{\tilde{p}_3}{d_3} \geq \cdots \]

Curve is concave due to (\# \# \#).
Consider condition (ii). For \( \alpha > 1 \), the function
\[ g(x) := x^\alpha \] is convex, thus
\[ \sum_i p_i^\alpha d_i^{1-\alpha} \geq (\sum_i (p_i^\alpha d_i^{1-\alpha})^{1-\alpha})^{\alpha-1} \]

**Def.** (Relative Rényi entropy).
For \( \alpha \in \mathbb{R} \setminus \{0, 1\} \), set for prob. vectors \( p, q \in \mathbb{R}^n \):
\[ D_\alpha (p \| q) := \frac{\text{sgn} \alpha}{\alpha-1} \log \sum_{i=1}^n p_i^\alpha q_i^{1-\alpha} \]

Furthermore,
\[ D_0 (p \| q) := \lim_{\alpha \to 0^+} D_\alpha (p \| q) = -\log \sum_{i=1}^n q_i \]
\[ D_1 (p \| q) := \lim_{\alpha \to 1} D_\alpha (p \| q) = \sum_{i=1}^n p_i (\log p_i - \log q_i) \]
standard relative entropy
\[ D_\infty (p \| q) := \lim_{\alpha \to \infty} D_\alpha (p \| q) = \log \max_i \frac{p_i}{q_i} \]
\[ D_{-\infty} (p \| q) := \lim_{\alpha \to -\infty} D_\alpha (p \| q) = D_\infty (q \| p) \]

**Thm.** (\( p, d \) > (\( p', d' \)) =)
\[ D_\alpha (p \| d) > D_\alpha (p' \| d') \quad \forall \alpha \]

Thermal operations decrease all relative-entropy distances to the Gibbs state.