- Sorry, exercise sheet 9 went online quite late.
  → can give back until Thursday, 17:00 (to me or one of the secretaries in Philosopheweg 19).
- Exercise sheet 10 will be the last one.

Recap last lecture:
- For block-diagonal states $[H, S] = 0$: can reduce to the classical case.
- Equivalent are, classically:
  - $p \xrightarrow{\text{thermal}} p'$, if initial and final Hamiltonians are such that the Gibbs states are $d$ and $d'$,
  - $(p, d) \succ (p', d')$, i.e. $\mathbb{E}_{\text{stochastic B with } Bp = p'}$ and $Bd = d'$.

- Thermal Lyapunov curve of $p$ everywhere on or above that of $p'$.

Sort such that $\frac{d_1}{d_1} \succ \frac{d_i}{d_i}$.

- for every convex function $g$,
  \[ \sum d_i g\left( \frac{p_i}{d_i} \right) \succ \sum d_i g\left( \frac{p_i'}{d_i'} \right). \]
  → starting point for Rényi divergences.
3.7. The Rényi divergence

Consider the condition for \( g(\xi) = \pm \xi^2 \)

**Def. (Relative Rényi entropies)**

If \( p, q \in \mathbb{R}^n \) are probability vectors, and \( \alpha \in \mathbb{R} \setminus \{0, 1\} \),

\[
D_{\alpha}(p || q) := \frac{\text{sgn} \alpha}{\alpha - 1} \log \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}.
\]

Furthermore,

\[
D_0(p || q) := \lim_{\alpha \to 0^+} D_{\alpha}(p || q) = -\log \sum_{i=1}^{n} q_i,
\]

\[
D_1(p || q) := \lim_{\alpha \to 1} D_{\alpha}(p || q) = \sum_{i=1}^{n} p_i (\log p_i - \log q_i),
\]

standard relative entropy \( D(p || q) \)

\[
D_{\infty}(p || q) := \lim_{\alpha \to \infty} D_{\alpha}(p || q) = \log \max_i \frac{p_i}{q_i},
\]

\[
D_{-\infty}(p || q) := \lim_{\alpha \to -\infty} D_{\alpha}(p || q) = D_0(q || p).
\]

Thus, if \((p, d) > (p', d')\) then \(D_{\alpha}(p || d) > D_{\alpha}(p' || d')\)

**Notation:**

\[
D_{\min} = D_0, \quad D = D_1, \quad D_{\max} = D_{\infty}
\]

\[
= D_{\min} \leq D \leq D_{\max}.
\]

**Caution:**

\[
H_{\min} = H_{\infty}, \quad H = H_1, \quad H_{\max} = H_0 \quad \text{(entropies)}
\]

\[
= H_{\min} \leq H \leq H_{\max}.
\]
3.8 Work of formation and the max - relative entropy

Suppose we want to create a system. How much work does that cost?

Start with energy eigenstate $|E\rangle$ of another Hamiltonian $H$.

Want $(|E\rangle, H) \xrightarrow{\text{thermal}} (\rho, H')$.

What's the smallest possible $E$?

Thermov Loewentanz case of $|E\rangle$ ($p_E = 1$, all other $p_i = 0$)

\[
\begin{align*}
\text{We only have to check the condition for the initial} & \\
\text{slope:} & \\
\frac{1}{\beta E} & \geq \frac{p_i}{y_i} = \max_i \frac{p_i}{y_i} = e^{D_{\infty}(p || y')} \tag{*}
\end{align*}
\]

\[
E \geq \frac{1}{\beta} D_{\infty}(p || y') - \frac{1}{\beta} \log Z.
\]
If energy levels of $H$ sufficiently dense, we can get (close to) equality.

Work of formation for thermal state:

$$\log Z = \log \sum_i e^{-\beta E_i} \geq \log e^{-\beta E_{\text{min}}} = -\frac{1}{\beta} \log Z \leq E_{\text{min}}$$

$E > E_{\text{min}} \geq -\frac{1}{\beta} \log Z,$

so (b) is automatically satisfied. Thermal states are "free!"

Lorentz cone of $f'$:

(* is equivalent to

$$D_\infty \left( \frac{1}{E(X|E)} \right) \geq D_\infty (p \| f')$$

$$\iff F_\infty \left( \frac{1}{E(X|E)} \right) - F_\infty (f') \geq F_\infty (p) - F_\infty (f')$$

where

$$F_\infty (p) = \frac{1}{\beta} D_\infty (p \| f') - \frac{1}{\beta} \log Z'$$

(if the system is $P = (p, \mathcal{H})$) with $f'$ Gibbs state.

"Max-free energy" $F_\infty := F_{\text{max}}$
3.9. Extractable work and the min-relative entropy.

Suppose we are given a system $P = (p, H')$ and want to create a system $(1E\rangle, H)$. What's the largest possible $E$ we can get?

Need "flat tail" $= 0$ entries of $P$.

\[
\text{green tail length} < \text{pink tail length}
\]

\[
1 - E = 1 - \frac{e^{-B E}}{2} \leq \sum_{i: p_i = 0} e^{-B E_i} = 1 - \sum_{i: p_i \neq 0} x_i^{i'}
\]

\[
= 1 - e^{-D_{o}(p||g')}
\]

\[
E \leq \frac{1}{B} D_{o}(p||g') - \frac{1}{B} \log Z \quad (***)
\]

If we choose $H$ suitably, we can achieve equality.
If \( p = y' \): \( D_o(p|y') = 0 \Rightarrow \)

\[ E \leq -\frac{1}{\beta} \log z \leq E^* \]

\( \Rightarrow \) Can either get the ground state, or no pure state at all.

\( \exists \) does not contain useful work.

\[ (**) \]

is equivalent to

\[ D_o(p|y') \geq D_o(1|\text{EXEL}, y) \]

\[ \Leftrightarrow F_0(p) - F_0(y') \geq F_0(1|\text{EXEL}) - F_0(y) \]

where \( F_0(p) := \frac{1}{\beta} D_o(p|y') - \frac{1}{\beta} \log z' \)

is the "min-free energy" \( F_0 := F_{\text{min}} \).

In general \( D_o \ll D_0 \)

\( \text{"} \text{D}_{\text{min}} \text{D}_{\text{max}} \) fundamental asymmetry

\( \Rightarrow \)

\[ E_{\text{extractable}} \leq \frac{1}{\beta} D_o(p|y') - \frac{1}{\beta} \log z \]

\[ \ll \frac{1}{\beta} D_0(p|y') - \frac{1}{\beta} \log z \leq E_{\text{formation}}. \]
3.10. Allowing finite error probability: 

**Smoothing**

Introduce "smoothed Rényi divergences":

\[ D_0^\varepsilon(p \| q) := \max_{p : D(p, p') \leq \varepsilon} D_0(p' \| q), \]

\[ D_\infty^\varepsilon(p \| q) := \min_{p' : D(p, p') \leq \varepsilon} D_\infty(p' \| q). \]

**Approximate work of formation:**

Want:

\[ (1E > H) \xrightarrow{\text{thermal}} (p', H') \text{ with } D(p, p') \leq \varepsilon. \]

Minimal work needed is therefore:

\[ W_\text{form}^\varepsilon = \min_{p' : D(p, p') \leq \varepsilon} \left( \frac{1}{\beta} D_\infty(p' \| q') - \frac{1}{\beta} \log 2 \right) \]

\[ = \mathbf{1} \frac{1}{\beta} D_\infty^\varepsilon(p \| q) - \frac{1}{\beta} \log 2 \]

**Approximate extractable work:**

Want \((p, H') \xrightarrow{\text{thermal}} (q, H) \text{ with } D(q, \text{1EXE1}) \leq \varepsilon. \)

Choose \(p'\) with \(D_0^\varepsilon(p \| q') = D_0(p' \| q'), \ D(p, p') \leq \varepsilon.\)

Can obtain \((\text{1EXE1}, H)\) by thermal op. \(\phi\) from \(p',\) where

\[ E \leq \frac{1}{\beta} D_0(p' \| q') - \frac{1}{\beta} \log 2 = \frac{1}{\beta} D_\infty^\varepsilon(p \| q') - \frac{1}{\beta} \log 2. \]

\[ = D(\phi(p), \text{1EXE1}) = D(\phi(p), \phi(p')) \leq D(p, p') \leq \varepsilon. \]
\[ W^e_{\text{ex}} \geq \frac{4}{\beta} D_0^e (p|\bar{y}) - \frac{4}{\beta} \log 2. \]

Exact expression is a bit harder to get; maybe homework.