Single-shot thermo, lecture 14

* Jakob gives back last corrected exercises.

Today (partly with projector): peripheral results / relation to majorization, single-shot entropies; current research.

4.2. Attempts of unification with fluctuation-dissipation theorems

Crooks' theorem:

- forward protocol: parameter $X(t)$ determines time-dependent Hamiltonian $H(X(t)) = H_t$. At $t = -T$: start in state $Y_{-T} = e^{-\beta H_{-T}/2}$. Interactions with heat bath. Stop at $t = T$ (possibly let thermalize to $Y_T$).

- reverse protocol: run backwards.

Repeat many times $\Rightarrow$ prob. density $P_{\text{forward}}(W)$ over required work $W$, $P_{\text{reverse}}(W)$ over gained work $W$.

Crooks:

$$ \frac{P_{\text{forward}}(W)}{P_{\text{reverse}}(-W)} = e^{\beta(W - \Delta F)} $$

$$ \Delta F = F(Y_T) - F(Y_{-T}) $$

Free energy difference.

$$ \int P_{\text{forward}}(W) e^{\beta W} dW = \int P_{\text{reverse}}(-W) e^{-\beta \Delta F} dW $$

$$ \langle e^{-\beta W} \rangle_{\text{forward}} = e^{-\beta \Delta F} $$

Jasnyshi's equality

Can be used to determine equilibrium quantity $\Delta F$ from out-of-equilibrium measurements.
prob. distribution over work (cf. Landauer erasure etc.)

Lemma: \[ \frac{1}{\beta} D(P_{\text{rev}}(-W) \| P_{\text{uld}}(W)) = \Delta F - \langle W \rangle_{\text{rev}} \]
where \( \langle W \rangle_{\text{rev}} = \) average work gain from reverse process.

Note: \( D(\cdot \| \cdot) \geq 0 \Rightarrow \langle W \rangle_{\text{rev}} \leq \Delta F \),
one obtains less work if \( P_{\text{rev}}(-W) \) and \( P_{\text{uld}}(W) \)
are more distinguishable (irreversibility).

\textbf{Lemma: } \( W_{\text{min}} := \) the least work that any reverse trial can output. Then
\[ \frac{1}{\beta} D_{\text{max}}(P_{\text{rev}}(-W) \| P_{\text{uld}}(W)) = \Delta F - W_{\text{min}} \]

Recall \( D_{\text{min}} \leq D \leq D_{\text{max}} \)

5. Majorization and entanglement: Nielsen's Theorem

Entangled quantum states

LOCC: local operations and classical communication.

→ cannot create entanglement.

But they can for example distinguish the Bell states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)$$

Local reduced states are $$S_A = S_B = \frac{1}{2} (1|0\rangle_0 \langle 0| + 1|1\rangle_1 \langle 1|$$.

But: Alice measures qubit (→ 0 as 1), sends result to Bob, Bob measures -> they know!

Question: Given $$|\psi\rangle, |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$$, can A and B transform $$|\psi\rangle$$ into $$|\phi\rangle$$ via LOCC?

(Intuition: only if $$|\psi\rangle$$ "less entangled than" $$|\phi\rangle").
Resource Theory of Entanglement

Free states: product states \( |\psi_A \rangle \otimes |\psi_B \rangle \)

Free things: LOCC

Resources: Entangled states

Every \( |\psi_{AB} \rangle \) can be written

\[
|\psi_{AB} \rangle = \sum_{i=1}^{d_A} \sqrt{a_i} |i_A \rangle \otimes |i_B \rangle
\]

\( \text{local CMN} \)

\( \Rightarrow \) both \( S_A \) and \( S_B \) have eigenvalues \( A_1, \ldots, A_d \).

Nielsen's Theorem: \( |\psi \rangle \rightarrow |\psi' \rangle \) by LOCC if and only if \( A < A' \), where \( A \) and \( A' \) are the Schmidt coefficients of \( |\psi \rangle \) and \( |\psi' \rangle \).

Example: \( |\psi_{AB} \rangle = |\psi \rangle \otimes |\psi' \rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \)

\( \Rightarrow A = (1, 0, \ldots) \) useless state

\( |\psi_{AB} \rangle = \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle) \)

\( \Rightarrow A' = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \) very valuable (Bell) state

\( \Rightarrow |\psi \rangle \rightarrow |\psi' \rangle \) but not the other way round.

- Resource th. of entanglement analogous, but "inverse" to res. th. of nonuniformity.
<table>
<thead>
<tr>
<th>res. H. of Entanglement</th>
<th>res. H. of Non-separability</th>
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<tbody>
<tr>
<td><strong>useless stale</strong></td>
<td></td>
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<tr>
<td>( \rho_{A} = \frac{1}{d}</td>
<td>\rho_{A} \otimes</td>
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<td>max. entangled state</td>
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<tr>
<td>with ( \rho = (\frac{1}{d}, \ldots, \frac{1}{d}) )</td>
<td>( q = (1, 0, \ldots, 0) )</td>
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<td>max. possible if...</td>
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<td>( \rho \rightarrow</td>
<td>\rho )</td>
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<td>( \rho_{A} \otimes</td>
<td>\rho_{A} \rightarrow S(\rho) )</td>
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<tr>
<td><strong>transition table</strong></td>
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| \( \frac{1}{n} \) \rightarrow S(\rho) \) | \( \frac{1}{n} \rightarrow \log 

\[ p \rightarrow \log \frac{1}{p} - N(p) \]

\[ \frac{1}{n} \rightarrow \frac{1}{\log q} - H(q) \]

\[ \Rightarrow \text{entanglement entropy} \ S(\rho) \]
gives asymptotically the number of "Bell pairs" 

\[ \frac{1}{\sqrt{2}} (|100\rangle + |111\rangle) \]

that can be distilled from \( \rho \) via LOCC.
(or are needed to form \( |\psi\rangle \))